On the energy estimates for second order homogeneous hyperbolic equations with Levi-type conditions [1]

Fumihiko Hirosawa (Yamaguchi University)

and

Bui Tang Bao Ngoc (Hanoi University of Technology)

We consider the following Cauchy problem of a second order homogeneous hyperbolic equation with variable coefficients:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\partial_x^2 + 2b(t)\partial_x\partial_t\right) u = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}, \end{cases} \tag{1}$$

where $a(t), b(t) \in C^m([0,\infty))$ $(m \geq 2)$ are real valued and satisfy the following strictly hyperbolic condition:

$$0 < c_0 \leq c(t) := \sqrt{a(t)^2 + b(t)^2} \leq c_1. \tag{2}$$

Here we introduce the following energy to the solution of (1):

$$E(t) := \frac{1}{4}\int_{\mathbb{R}} \left(|\partial_t u(t,x) + (b(t)+c(t))\partial_x u(t,x)|^2 + |\partial_t u(t,x) + (b(t)-c(t))\partial_x u(t,x)|^2\right) dx. \tag{3}$$

If the coefficients are constants, then the energy conservation $E(t) \equiv E(0)$ is valid. However, we cannot expect such a property for variable coefficients; thus we introduce the following property of an equivalence of the energy with respect to t:

$$C^{-1}E(0) \leq E(t) \leq CE(0), \tag{GEC}$$

which is called the *generalized energy conservation*, where $C > 1$ is a constant.

If $b(t) = 0$, then the equation of (1) is a wave equation with a variable propagation speed, and $a'(t)$ describes the oscillating speed of it. Trivially, we see that (GEC) is valid if $a' \in L^1(\mathbb{R}_+)$ though $b(t) \neq 0$. However, it is not clear whether (GEC) holds or not if $a' \notin L^1(\mathbb{R}_+)$. Actually, (GEC) is not true in general; indeed, for $b(t) = 0$ an example of $a(t)$ is constructed in [6]. The main purpose of our research is to have some conditions to the coefficients which provide (GEC) taking account of the C^m regularity of the coefficients. In particular, we focus the conditions between $a(t)$ and $b(t)$, which give the same conclusion of [3] considering (1) with $b(t) = 0$.

Let us introduce the following conditions to the coefficients:

- *Stabilization condition*: there exist the means a_∞ and b_∞ of $a(t)$ and $b(t)$ on $\mathbb{R}_+$ such that

$$\int_0^t (|a(s) - a_\infty| + |b(s) - b_\infty|)\, ds \leq C_0(1+t)^\alpha \quad \text{for} \quad \alpha \in [0,1). \tag{4}$$

- *Control of the oscillations*:

$$|a^{(k)}(t)| + |b^{(k)}(t)| \leq Cake(1+t)^{-k\beta} \quad \text{for} \quad \beta \in [0,1] \quad (k = 1, \cdots, m). \tag{5}$$

Remark 1. The stabilization condition (4) is trivial for $\alpha = 1$ since (2) is valid. The condition of control of the oscillations (5) with $\beta > 1$ with $k = 1$ gives $a'(t) \in L^1(\mathbb{R}_+)$.

[1] The 26th Matsuyama camp, "Dedicated to the 60th Birthday of Professor Morimoto", January 6-9, 2011, Kyoto.

Let us recall the following result in [3] for $b(t) = 0$:

Theorem 1 ([3])**.** *Let $b(t) = 0$ and $m \geq 2$. If $a(t)$ satisfies (2), (4) and (5) for*

$$\beta = \beta_m := \alpha + \frac{1-\alpha}{m}, \tag{6}$$

then (GEC) is valid.

REMARK 2. The restriction to the order of $a'(t)$ in (5) is weaker as β larger, which is realized as m larger. That is, faster oscillation to the coefficient is possible to be permissible for (GEC) as the coefficient is smoother. Here we underline that we have a benefit by the choice of larger m only for $\alpha < 1$; thus the stabilization property (4) is essential.

It may be natural that we expect the same conclusion of Theorem 1 for $b(t) \neq 0$. However, we see from the analogy of the result in [4] that such an expectation is not valid to the general model (1) with $b(t) \neq 0$, because an interaction between the oscillating coefficients $a(t)$ and $b(t)$ gives a bad effect for (GEC). On the other hand, the result in [5] hints us that the following condition between $a(t)$ and $b(t)$:

$$\sup_t \left\{ \left| \int_0^t \frac{b'(s)}{c(s)}\, ds \right| \right\} \leq C, \tag{L1}$$

which is called the *C^1-type Levi condition*, is possible to invalidate the bad effect from the interactions of the oscillating coefficients. Indeed, we have the following theorem:

Theorem 2 (C^3 coefficients [1])**.** *Let $m = 2$ or 3, $\alpha \in [0,1)$ and $\beta = \beta_m$. Assume that $a, b \in C^m([0,\infty))$ satisfy (2), (4) and (5). If the C^1-type Levi condition (L1) holds, then the generalized energy conservation (GEC) is valid.*

REMARK 3. (L1) is true if $a(t)$ is represented by $a(t) = \phi(b(t))$ with a positive C^1 function ϕ.

REMARK 4. Actually, under the assumption (L1) one can prove (GEC) for $m = 2$ and $\beta (= \beta_b) = 1$ (see [2, 7], which consider more general hyperbolic systems and L^p-L^q type decay estimates).

We cannot have the same conclusion as Theorem 2 for $m \geq 4$. However, we have the following theorem for (GEC) with $m = 4, 5$ if we additionally suppose the *C^2-type Levi condition*:

Theorem 3 (C^5 coefficients [1])**.** *Let $m = 4$ or 5, $\alpha \in [0,1)$ and $\beta = \beta_m$, where β_m is defined by (6). Assume that $a, b \in C^m([0,\infty))$ satisfy (2), (4) and (5). If the C^1-type Levi condition (L1) and the C^2-type Levi condition:*

$$\sup_t \left\{ (1+t)^{2\alpha} \left| \int_0^t \frac{c(s)\left(b'(s)c''(s) - b''(s)c'(s)\right) - b'(s)\left((b'(s))^2 - (c'(s))^2\right)}{c(s)^5}\, ds \right| \right\} \leq C \tag{L2}$$

hold, then the generalized energy conservation (GEC) is valid.

It is a natural observation that Theorem 2 and 3 may be generalized for $m = 6, 7$, $m = 8, 9$, and so on under some *C^k-type Levi conditions* with $k = 3, 4, \cdots$. Actually, it is true in a certain sense, but the representations of the corresponding C^k-type Levi conditions are very complicate.

References

[1] T. B. N. Bui, F. Hirosawa, On the energy estimates for second order hyperbolic equations with time dependent coefficients. Preprint.

[2] M. D'Abbicco, S. Luccente, G. Taglialatela, L^p-L^q estimates for regularly hyperbolic systems. *Adv. Differential Equations* **14** (2009), 801–834.

[3] F. Hirosawa, On the asymptotic behavior of the energy for the wave equations with time depending coefficients. *Math. Ann.* **339** (2007), 819–839.

[4] F. Hirosawa, M. Reissig, About the optimality of oscillations in non-Lipschitz coefficients for strictly hyperbolic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)* **3** (2004), 589–608.

[5] F. Hirosawa, M. Reissig, Levi condition for hyperbolic equations with oscillating coefficients. *J. Differential Equations.* **223** (2006), 329–350.

[6] M. Reissig, J. Smith, L^p-L^q estimate for wave equation with bounded time dependent coefficient. *Hokkaido Math. J.* **34** (2005), 541–586.

[7] M. Ruzhansky, J. Wirth, Corrigendum to "dispersive estimates for T-dependent hyperbolic systems". *Rend. Sem. Mat. Univ. Politec. Torino* **68** (2010), 339–349.