Energy estimates for wave equations with time dependent coefficients

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We consider the following Cauchy problem for a wave equation with time dependent propagation speed $a = a(t)$:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\Delta\right) u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (u(0,x), (\partial_t u)(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where we suppose that $a \in C^1([0,\infty))$, and $a_0 \le a(t) \le a_1$ for positive constants a_0 and a_1. Then the total energy of (1) at t is given by

$$E(t) = \frac{1}{2}\left(a(t)^2\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\partial_t u(t,\cdot)\|_{L^2}^2\right). \tag{2}$$

If $a(t)$ is a constant, then the energy conservation $E(t) \equiv E(0)$ holds. However, such a property does not hold in general for variable variable propagation speeds; thus we consider the following energy estimates:

$$\eta(t)^{-1}E(0) \le E(t) \le \eta(t)E(t) \quad (t \to \infty), \tag{3}$$

where the error $\eta(t)$ is monotone increasing and satisfies $\eta(t) > 1$. In particular, we call the estimate (3) with $\eta(t) = C$ *generalized energy conservation* (=GEC), where C is a positive constant.

If $|a'(t)| \le C$, then by the inequalities

$$-\frac{2|a'(t)|}{a(t)}E(t) \le E'(t) = a'(t)a(t)\|\nabla u(t,\cdot)\|_{L^2}^2 \le \frac{2|a'(t)|}{a(t)}E(t)$$

we have (3) with $\eta(t) = e^{Ct}$. Moreover, if $|a'(t)| \le C(1+t)^{-\beta}$ for a $\beta \ge 0$, then we have (3) with $\eta(t) = e^{Ct^{-\beta+1}}$ for $\beta < 1$, $\eta(t) = t^C$ for $\beta = 1$, and $\eta(t) = C$ for $\beta > 1$; thus faster decaying $|a'(t)|$ contribute to the stabilization of the energy.

If $a \in C^2([0,\infty))$, then the order of $\eta(t)$ can be improved as follows:

Theorem 1 ([5])**.** *If $a \in C^2([0,\infty))$ satisfies*

$$|a^{(k)}(t)| \le C_k(1+t)^{-k} \tag{4}$$

for $k = 1, 2$, then GEC is valid.

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Moreover, if $a \in C^m([0,\infty))$ $(m \ge 2)$, then the order of $\eta(t)$ can be improved corresponding to m under the following assumption, which is called the *stabilization property*:

$$\int_0^t |a(s) - a_\infty|\, ds = O(t^\alpha) \quad (0 \le \alpha < 1), \tag{5}$$

where $a_\infty = \lim_{t\to\infty} \int_0^t a(s)\, ds/t$.

Theorem 2 ([2]). *If $a \in C^m([0,\infty))$ $(m \ge 2)$ satisfies* (5) *for a $\alpha \in [0,1)$ and*

$$|a^{(k)}(t)| \le C_k (1+t)^{-k\beta} \tag{6}$$

for $k = 1, \cdots, m$, then we have (3) *with $\eta(t) = \exp(Ct^{\sigma_m})$, where*

$$\sigma_m = \max\left\{0, \alpha - \beta + \frac{1-\alpha}{m}\right\}. \tag{7}$$

Let us consider the limit case of Theorem 2 as $m \to \infty$ to introduce the Gevrey class γ^ν $(\nu > 1)$:

$$\gamma^\nu = \left\{ f(t) \in C^\infty([0,\infty))\,;\, |f^{(k)}(t)| \le C\rho^{-k} k!^\nu,\ \exists \rho > 0 \right\}.$$

For $a \in \gamma^\nu$ and a non-negative constant δ we introduce the following conditions:

$$|a^{(k)}(t)| \le Ck!^\nu \left((1+t)^\alpha (\log(e+t))^\delta \right)^{-k} \quad (k = 1, 2, \cdots). \tag{8}$$

Then we have the following result, which gives precise estimates of (3) for $m = \infty$ and $\alpha = \beta$:

Theorem 3 ([3]). *If $a \in \gamma^\nu$ satisfies* (5) *and* (8) *for a $\alpha \in [0,1)$, then we have* (3) *with $\eta(t) = \exp(C(\log t)^\sigma)$, where*

$$\sigma = \max\{0, \nu - \delta\}. \tag{9}$$

Summarizing Theorem 2 and Theorem 3, we have the following table for the relations between the smoothness of $a(t)$ and the order of $\eta(t)$ under the assumptions (5) and (8) with $\delta = 0$:

$a(t)$	C^1	C^m $(m \ge 2)$	C^∞	γ^ν $(\nu > 1)$
$\eta(t)$	$\exp(Ct^{1-\alpha})$	$\exp(Ct^{\frac{1-\alpha}{m}})$	$\exp(Ct^\varepsilon)$	$\exp(C(\log t)^\nu)$

where ε is an arbitrarily positive constant.

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