Energy estimates for wave equations with time dependent coefficients

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1. Introduction

Consider the following Cauchy problem and the total energy:

$$\begin{cases} (\partial_t^2 - a(t)^2 \Delta)\, u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

$0 < a_0 \le a(t) \le a_1$, $a(t) \in C^1([0,\infty))$.

$$E(t) = \frac{1}{2} \left(a(t)^2 \left\| \nabla u(t,\cdot) \right\|^2 + \left\| \partial_t u(t,\cdot) \right\|^2 \right) \tag{2}$$

If $a(t)$ is a constant, then the energy conservation (EC) is valid:

$$E(t) \equiv E(0), \tag{EC}$$

however, (EC) is not valid if $a(t)$ is not a constant.

Generally, we only expect the following estimates:

$$\eta(t)^{-1} E(0) \le E(t) \le \eta(t) E(0) \quad (\eta(t) > 1). \tag{3}$$

$\eta(t) \equiv C \Leftrightarrow$ GEC(=Generalized Energy Conservation).

Monotone increasing or decreasing coefficients

$$E'(t) = a'(t)a(t)\|\nabla u(t)\|^2$$

$$a'(t) \le 0 \;\Rightarrow\; \begin{cases} E'(t) \le 0 \;\Rightarrow\; E(t) \le E(0) \\ \\ E'(t) \ge \dfrac{2a'(t)}{a(t)} E(t) \\ \quad \Rightarrow\; E(t) \ge \exp\left(2\log \dfrac{a(t)}{a(0)}\right) E(0) = \dfrac{a(t)^2}{a(0)^2} E(0) \end{cases}$$

$$\begin{cases} a'(t) \le 0 \;\Rightarrow\; E(0) \le E(t) \le \frac{a_1^2}{a_0^2} E(0) \\ a'(t) \ge 0 \;\Rightarrow\; \frac{a_0^2}{a_1^2} E(0) \le E(t) \le E(0) \end{cases}$$

Theorem 0.1. *If $a(t)$ is a monotone increasing or decreasing functions, then GEC is valid.*

Oscillating coefficients

$$-\frac{2|a'(t)|}{a(t)}E(t) \le E'(t) \le \frac{2|a'(t)|}{a(t)}E(t)$$

$$\Downarrow$$

$$\exp\left(-\frac{2}{a_0}\int_0^t |a'(s)|ds\right)E(0) \le E(t) \le \exp\left(\frac{2}{a_0}\int_0^t |a'(s)|ds\right)E(0)$$

Theorem 0.2. *If* $a(t) \in C^1([0,\infty))$ *satisfies*

$$|a'(t)| \le C(1+t)^{-\beta} \quad (\beta \ge 0),$$

then (3) *is valid for*

$$\eta(t) = \begin{cases} \exp\left(Ct^{-\beta+1}\right) & (\beta < 1), \\ t^C & (\beta = 1), \\ C & (\beta > 1). \end{cases}$$

Periodic coefficients

Theorem 0.3. *If* $\boldsymbol{a(t)}$ *is positive, periodic and non-constant, then for any* $\boldsymbol{\varepsilon > 0}$ *the following uniform estimates are not valid in general*:

$$\boldsymbol{E(t_1) \le E(t_0) \exp\left(C\left(t_1^{1-\varepsilon} - t_0^{1-\varepsilon}\right)\right) \quad 0 < \forall t_0 < \forall t_1 < \infty.}$$

Example. $\boldsymbol{a(t) = 2 + \cos t}$

$$\boldsymbol{\limsup_{t\to\infty}\{|a'(t)|\} = 1 \;\Rightarrow\; e^{-Ct}E(0) \le E(t) \le e^{Ct}E(0)}$$

Observations

- Can we derive some cancellation of the energy due to the oscillating coefficient?

$$a'(t) < 0 \Rightarrow E'(t) \leq 0, \;\; a'(t) > 0 \Rightarrow E'(t) \geq 0.$$

- The order of error $\eta(t)$ is described by the number of oscillation:

$$\int_0^t |a'(s)|ds.$$

- L^1 property of $a'(t)$ concludes GEC; $\eta(t) = 1$.

$a'(t) \notin L^1$

Theorem 1([5]). *If* $a(t) \in C^2([0,\infty))$ *satisfies*:

$$|a'(t)| \le C_1(1+t)^{-1}, \quad |a''(t)| \le C_2(1+t)^{-2},$$

then GEC is valid.

Remark. $a(t) = 2 + \cos(\omega(t))$, $\omega(0) = 0$, $\omega'(t) > 0$.

$$a' \in L^1 \Leftrightarrow \int_0^\infty \omega'(s)ds = \lim_{t\to\infty} \omega(t) < \infty$$

$\Leftrightarrow$ finite oscillation.

$$\omega(t) = \log(1+t) \Rightarrow \begin{cases} |a^{(k)}(t)| \le C_k(1+t)^{-k} \ (k=1,2); \\ a' \notin L^1. \end{cases}$$

- GEC can be valid for infinitely oscillating coefficient.
- Cancellation of the energy is realized.

Stabilization property and C^m property

$$a \in C^m \ (m \geq 2), \ \lim_{t \to \infty} \frac{1}{t} \int_0^t a(s) ds = \exists a_\infty.$$

We introduce the stabilization property with $\alpha \in [0, 1)$ and the C^m property with $\beta \in [0, 1)$ by

$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \ \ (t \to \infty) \qquad (5)$$

(*Stabilization property*)

$$|a^{(k)}(t)| \leq C_k (1+t)^{-k\beta} \ \ (k = 1, \cdots, m) \qquad (6)$$

(C^m *property*)

Theorem 2([2]). *If* **(5)** *and* **(6)** *are valid, then* **(3)** *holds for*

$$\boldsymbol{\eta(t) = \exp\left(C(1+t)^{\sigma_m}\right),}$$

$$\boldsymbol{\sigma_m = \max\left\{0,\ \alpha - \beta + \frac{1-\alpha}{m}\right\}.}$$

Remark.

- $\boldsymbol{\sigma_m}$ is monotone decreasing with respect to $\boldsymbol{m}$ and $\boldsymbol{\beta}$, and monotone increasing with respect to $\boldsymbol{\alpha}$.
- GEC holds if $\boldsymbol{\beta \geq \alpha + \frac{1-\alpha}{m}}$.
- The estimates (E) cannot be improved for $\boldsymbol{\beta < \alpha}$.
- $\boldsymbol{\alpha = \beta}$ is the critical case for GEC.

Orders of oscillating speed $|a'(t)|$ and error $\eta(t)$

$$|a'(t)| \le C(1+t)^{-\beta}$$

$$\eta(t)^{-1}E(0) \le E(t) \le \eta(t)E(0)$$

$$\int_0^t |a(s)-a_\infty|ds \le C(1+t)^{\alpha},\ \eta(t)=\exp(C(1+t)^{\sigma_m})$$

Regularity of a

Singular ⟷ Regular

m	1	2	$\cdots$	m	$\cdots$	∞
σ_m	$1-\beta$	$\frac{\alpha+1}{2}-\beta$	$\cdots$	$\alpha-\beta+\frac{1-\alpha}{m}$	$\cdots$	$\alpha-\beta$

Unstable ⟷ Stable

Behavior of $E(t)$

$(0 \le \alpha \le \beta < 1)$

2. Main results

We have the following questions from Theorem 2 as $m \to \infty$:

- $\alpha < \beta \Rightarrow \exists m \in \mathbb{N}$ s.t. GEC holds. Does GEC hold for $\alpha = \beta$?
- $\alpha = \beta \Rightarrow \eta(t) = \exp\left(C(1+t)^{\frac{1-\alpha}{m}}\right)$

  What about the order of $\eta(t)$ in the limit case $m = \infty$?

We consider such problems to introduce the Gevrey class of $a(t)$.

$$a(t) \in \gamma_\rho^\nu \Leftrightarrow |a^{(k)}(t)| \le C\rho^{-k}k!^\nu \quad (\nu > 1,\ \rho > 0)$$

$$C^\omega \subset \gamma_\rho^\nu \subset C^\infty$$

We consider the estimates (3) near the critical case $\alpha = \beta$ to introduce the following conditions:

$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \quad (t \to \infty) \tag{5}$$

$$|a^{(k)}(t)| \le C k!^\nu \left((1+t)^\alpha (\log(e+t))^\delta \right)^{-k} \quad (k \in \mathbb{N},\ \delta \ge 0) \tag{8}$$

$\beta \to \alpha$

$$|a^{(k)}(t)| \le C_k (1+t)^{-k\beta} \quad (k \in \mathbb{N},\ \alpha < \beta)$$

Theorem 3([3]). *If* **(5)** *and* **(8)** *are valid, then* **(3)** *holds for*

$$\eta(t) = \exp\left(C (\log(e+t))^\sigma \right), \quad \sigma = \max\{0,\ \nu - \delta\}.$$

Theorem 3([3]). *If* **(5)** *and* **(8)** *are valid, then* **(3)** *holds for*

$$\eta(t) = \exp\left(C\left(\log(e+t)\right)^{\sigma}\right), \quad \sigma = \max\{0, \nu-\delta\}.$$

$$\eta(t)^{-1}E(0) \le E(t) \le \eta(t)E(0) \tag{3}$$

$$\int_0^t |a(s)-a_\infty|ds = O(t^\alpha) \quad (t\to\infty) \tag{5}$$

$$|a^{(k)}(t)| \le Ck!^{\nu}\left((1+t)^{\alpha}\left(\log(e+t)\right)^{\delta}\right)^{-k} \tag{8}$$

Regularity of a

Singular ⟷ **Regular**

ν	∞	$\cdots$	ν	$\cdots$	1
σ	∞	$\cdots$	$\nu-\delta$	$\cdots$	$1-\delta$

Unstable ⟷ **Stable**

Behavior of $E(t)$

$(\nu \ge \delta)$

Summary

$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha)$$

$$|a^{(k)}(t)| \le C_k (1+t)^{-k\alpha} \ (k = 1, \cdots, m)$$

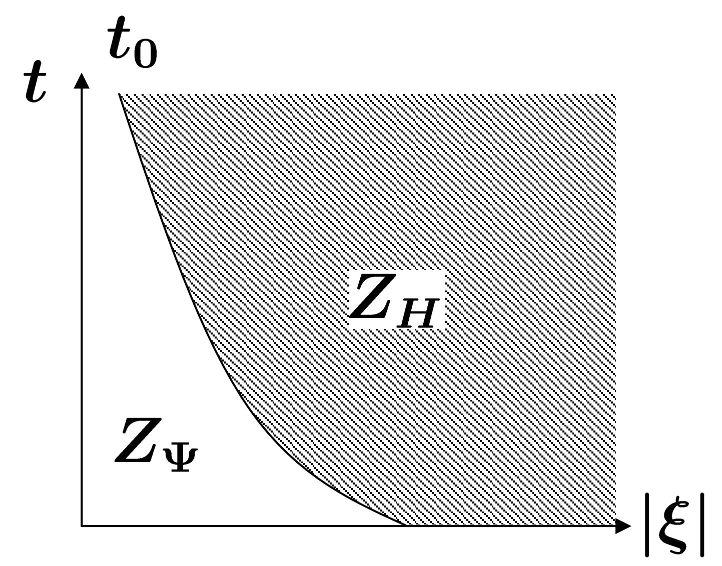
$$\eta(t)^{-1} E(0) \le E(t) \le \eta(t) E(0)$$

	$m < \infty$		$m = \infty$	$C_k = Ck!^\nu$	
$a(t)$	C^1	C^m	C^∞	γ^ν	C^ω
$\eta(t)$	$\exp(Ct^{1-\alpha})$	$\exp(Ct^{\frac{1-\alpha}{m}})$	$\exp(Ct^\varepsilon)$	$\exp(C(\log t)^\nu)$	t^C

$(0 \le \alpha < 1, \nu > 1)$

3. Keys of the proof

- Refined diagonalization
- Division of infinitely many zones
- Algebra of the Gevrey functions for symbol calculus



$$(\partial_t^2 + a(t)^2|\xi|^2)\, v(t,\xi) = 0$$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1, \quad V_1 = \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix} = \begin{pmatrix} \partial_t v + ia|\xi|v \\ \partial_t v - ia|\xi|v \end{pmatrix}$$

$$\Phi_1 = \begin{pmatrix} \frac{a'}{2a} + ia|\xi| & 0 \\ 0 & \frac{a'}{2a} - ia|\xi| \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -\frac{a'}{2a} \\ -\frac{a'}{2a} & 0 \end{pmatrix}$$

$$\boxed{\partial_t V_1 = (\Phi_1 + B_1)V_1} \quad \Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \overline{b_1} \\ b_1 & 0 \end{pmatrix}$$

$$V_2 = M_1^{-1} V_1, \quad M_1 = \begin{pmatrix} 1 & \overline{\delta_1} \\ \delta_1 & 1 \end{pmatrix}, \quad \delta_1 = \frac{-ib_1}{2\phi_{1,\Im}}$$

$$\boxed{|\delta_1| = \frac{|\frac{a'}{2a}|}{2a|\xi|} \leq \frac{C_1(1+t)^{\alpha-\beta}}{4a_0^2 N} \leq \frac{C_1}{4a_0^2 N} \leq \frac{1}{2} \qquad (\alpha \leq \beta, \quad N \gg 1)}$$

$$\boxed{\partial_t V_2 = (\Phi_2 + B_2)V_2} \quad \Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \overline{b_2} \\ b_2 & 0 \end{pmatrix}$$

$$\phi_{2,\Re} = \frac{1}{2}\partial_t \left(\log \left(\frac{a}{1 - |\delta_1|^2} \right) \right), \quad \phi_{2,\Im} = a|\xi| - \frac{2|\delta_1|^2}{1 - |\delta_1|^2}$$

$$\partial_t V_j = (\Phi_j + B_j) V_j$$

$$V_j = \begin{pmatrix} v_j \\ \overline{v_j} \end{pmatrix}, \Phi_j = \begin{pmatrix} \phi_j & 0 \\ 0 & \overline{\phi_j} \end{pmatrix}, \; B_j = \begin{pmatrix} 0 & \overline{b_j} \\ b_j & 0 \end{pmatrix}$$

$$V_{j+1} = M_j^{-1} V_j, \quad M_j = \begin{pmatrix} 1 & \overline{\delta_j} \\ \delta_j & 1 \end{pmatrix}, \quad \delta_j = \frac{-ib_j}{2\phi_{j,\Im}}$$

$$\partial_t V_{j+1} = (\Phi_{j+1} + B_{j+1}) V_{j+1}$$

$$\begin{cases} \phi_{j+1,\Re} = \dfrac{1}{2}\partial_t \left(\log \left(\dfrac{a}{\prod_{k=1}^{j} (1 - |\delta_k|^2)} \right) \right) \\[2ex] \phi_{j+1,\Im} = a|\xi| + \displaystyle\sum_{k=1}^{j} \dfrac{-2|\delta_k|^2 \phi_{k,\Im} + \Im\{\delta_k' \overline{\delta_k}\}}{1 - |\delta_k|^2} \\[2ex] b_{j+1} = \dfrac{b_j |\delta_j|^2 - \delta_j'}{1 - |\delta_j|^2} \qquad (j = 0, \cdots, m-1) \end{cases}$$

$$|V_m(t,\xi)|^2 \begin{cases} \le |V_m(t_0,\xi)|^2 \exp\left(2\int_{t_0}^t (\phi_{m,\Re} + |b_m|)\ ds\right) \\ \ge |V_m(t_0,\xi)|^2 \exp\left(2\int_{t_0}^t (\phi_{m,\Re} - |b_m|)\ ds\right) \end{cases}$$

$$|V_m(t,\xi)|^2 \simeq |V_1(t_0,\xi)|^2 \ \ (|M_k - I| \ll 1, \ \ k = 1, 2, \cdots, m)$$

$$|b_k(t,\xi)| \le C_k|\xi|^{-k+1}(1+t)^{-\beta k} \ \ (k = 1, 2, \cdots, m)$$

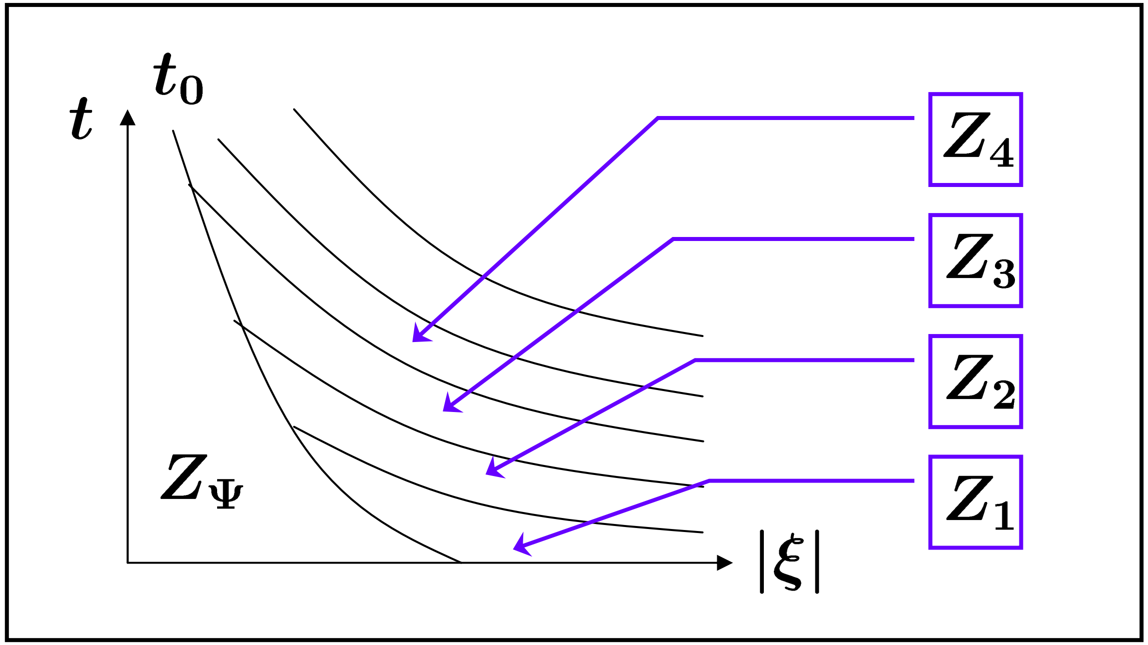
$$\begin{aligned} \int_0^t |\xi|^{-k+1}(1+s)^{-\beta k} ds &= \frac{1}{\beta k - 1}|\xi|^{-k+1}(1+t_0)^{-\beta k+1} \\ &= \frac{N^{-k+1}}{\beta k - 1}(1+t_0)^{-k(\beta-\alpha)+1-\alpha} \end{aligned}$$

The estimates of $|V_1|$ is improved by diagonalization procedures due to M_k.

- Division of infinitely many zones

$$Z_k = \{(t,\xi)\ ;\ t_{k-1} \le t \le t_k\}\,,$$

$$(1+t_k)^{\alpha}\,(\log(e+t_k))^{\delta}\,|\xi| = (k+1)^{\nu}$$



- Algebra of the Gevrey functions

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k!(n-k)!}{n!}\right)^{\nu} \le C \quad \Leftrightarrow \quad \nu > 1$$