Wave equations with time depending propagation speed

Fumihiko HIROSAWA [1]

What happens if the tension of a vibrating string is changed with respect to time?

Such a phenomenon is described by a partial differential equation, which is called a wave equation, and the solution is represented explicitly by an application of Fourier series if the tension is a constant. However, it is not easy to see the behavior of the solution to such kind of problem for non-constant tension, because the reduced ordinary differential equations from the wave equation are variable coefficients. Indeed, our problem is described by the following initial boundary value problem of a wave equation:

$$\begin{cases} (\partial_t^2 - a(t)^2\partial_x^2)u(t,x) = 0 & (t,x) \in \mathbf{R}_+ \times [-L,L], \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)) & x \in [-L,L], \\ u(t,-L) = u(t,L) = 0 & t \in \mathbf{R}_+, \end{cases} \tag{1}$$

where $a(t)$ is the propagation speed, which is determined by the tension of the string, satisfies $a_0 \le a(t) \le a_1$ with some positive constants a_0 and a_1. Then the total energy of the string at the time t is given by

$$E(t) = \frac{1}{2}a(t)^2 \int_{-L}^{L} |\partial_x u(t,x)|^2 dx + \frac{1}{2}\int_{-L}^{L} |\partial_t u(t,x)|^2 dx. \tag{2}$$

If $a(t)$ is a constant, then the energy conservation law: $E(t) \equiv E(0)$ holds. However, such a property does not hold in general for variable propagation speeds.

The main purpose of my talk is to derive the properties of the coefficient for a small perturbation of the constant coefficient.

[1]Department of Mathematical Sciences, Faculty of Science, Yamaguchi University, Yamaguchi 753-8512, Japan, hirosawa@yamaguchi-u.ac.jp