Wave equations with time depending propagation speed

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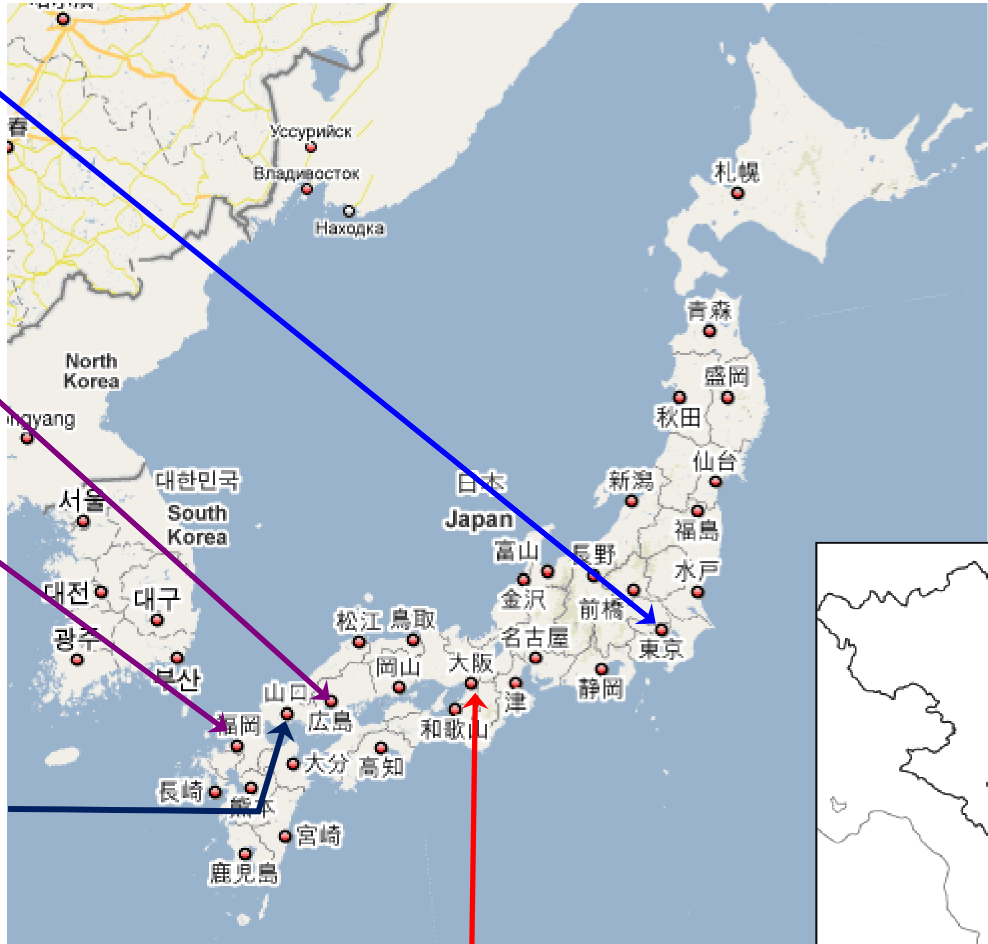

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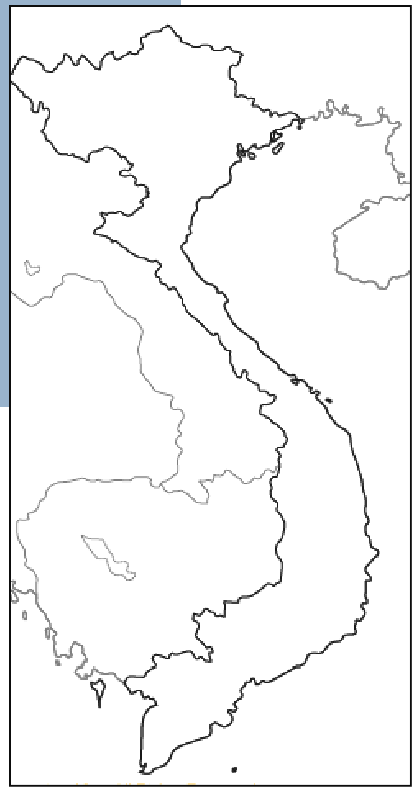
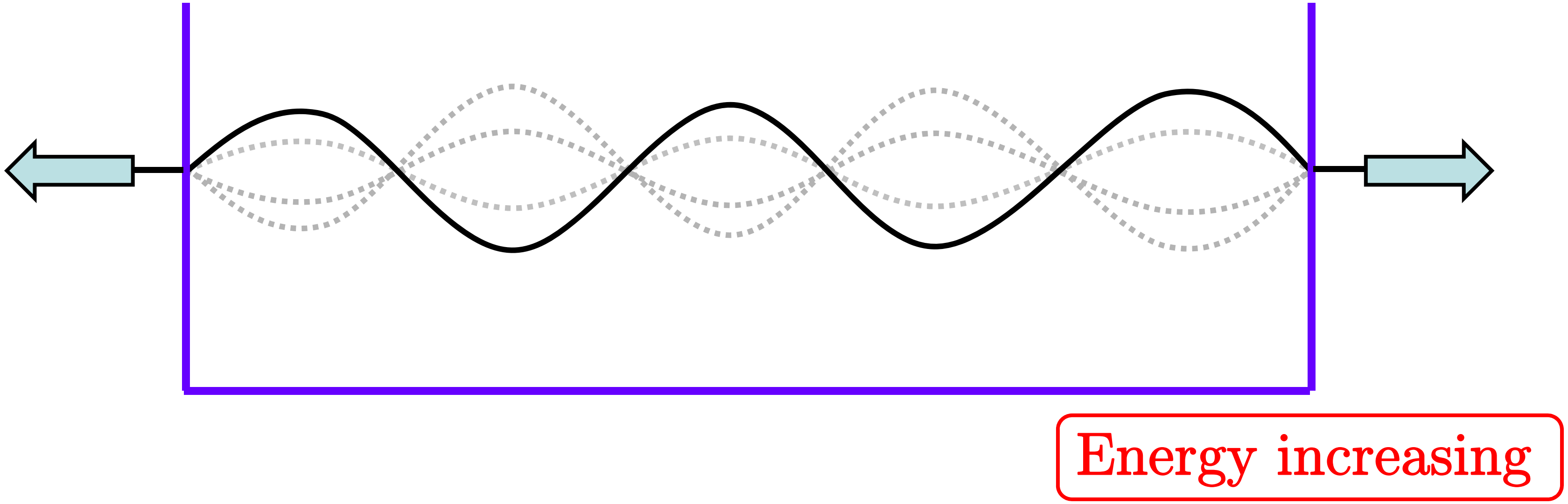
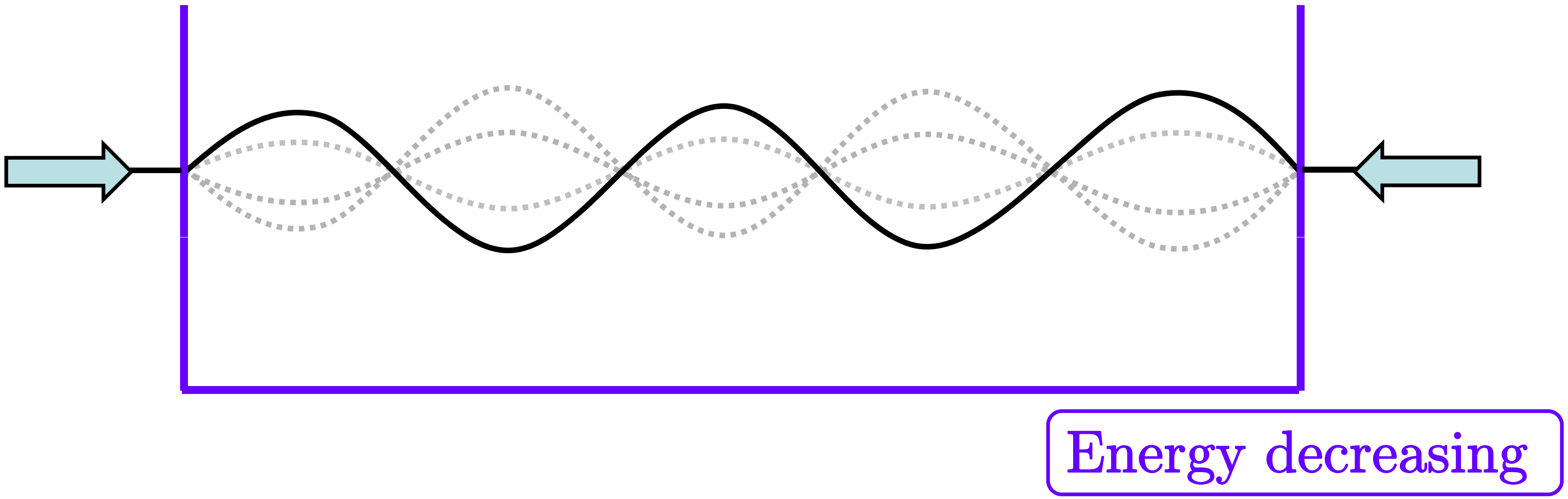
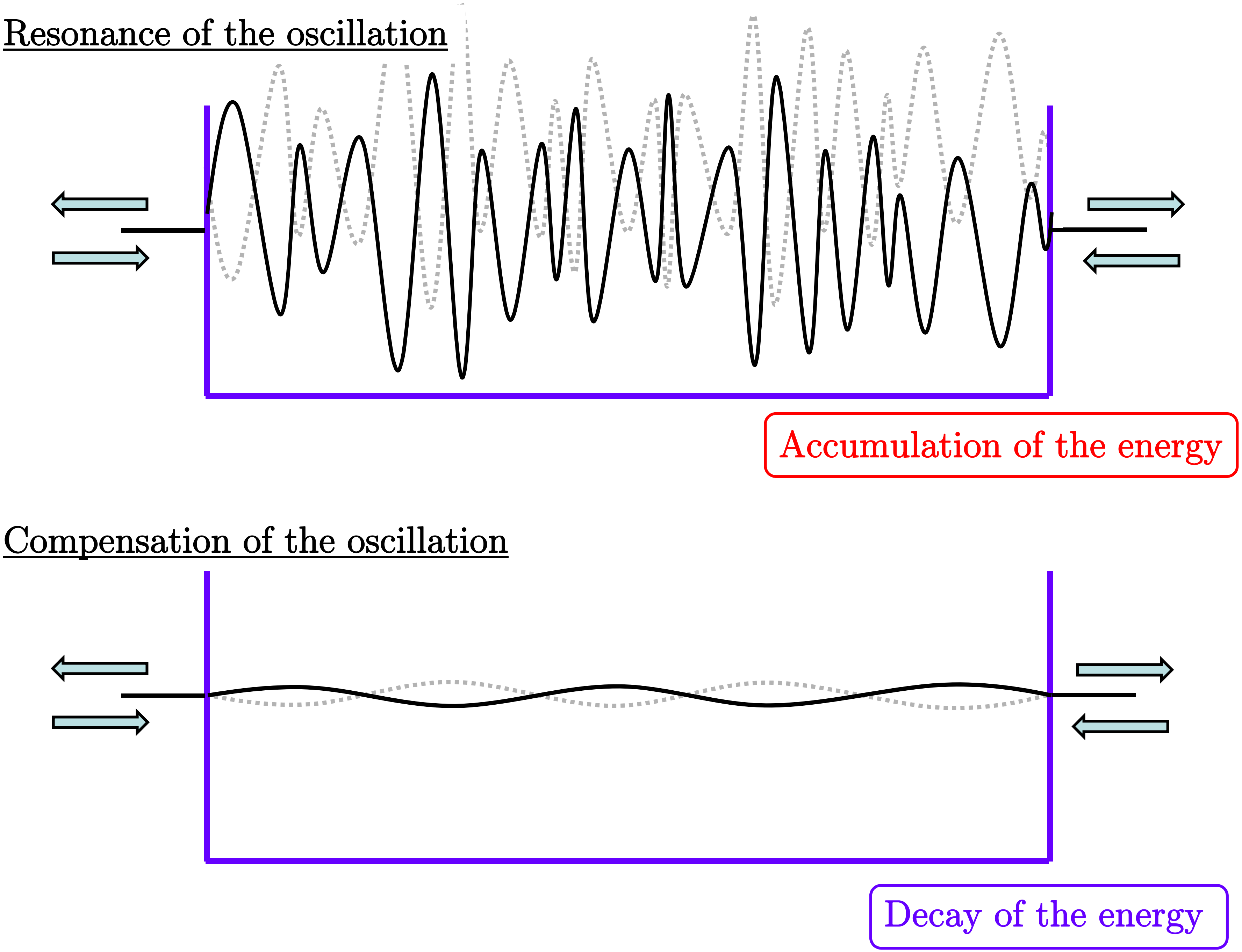


1. Introduction

Vibrating string with variational tension






Resonance of the oscillation
Accumulation of the energy
Compensation of the oscillation
Decay of the energy

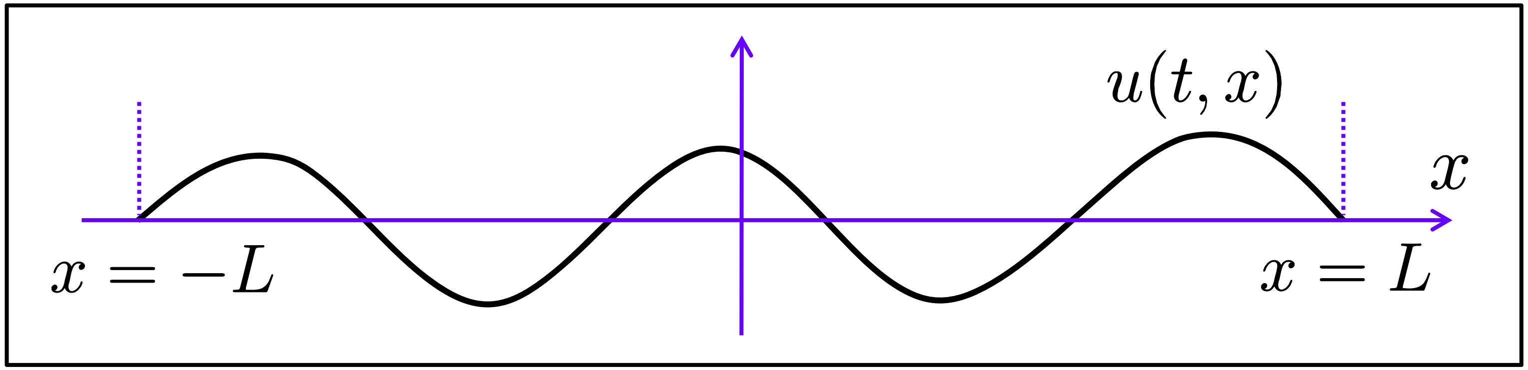
Question.

What conditions to the variational tension do provide a stabilization of the energy?

Model equation

$$(1)\quad \begin{cases} (\partial_t^2 - a(t)^2\partial_x^2)\, u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times [-L,L] \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in [-L,L] \\ u(t,-L) = u(t,L) = 0, & t \in \mathbb{R}_+ \end{cases}$$

$0 < a_0 \le a(t) \le a_1$, $a(t) \in C^1(\mathbb{R}_+)$: propagation speed



Total energy

$$(2)\quad E(t) = \frac{1}{2}a(t)^2 \int_{-L}^{L} |\partial_x u(t,x)|^2 dx + \frac{1}{2}\int_{-L}^{L} |\partial_t u(t,x)|^2 dx$$

Energy conservation for constant propagation speed

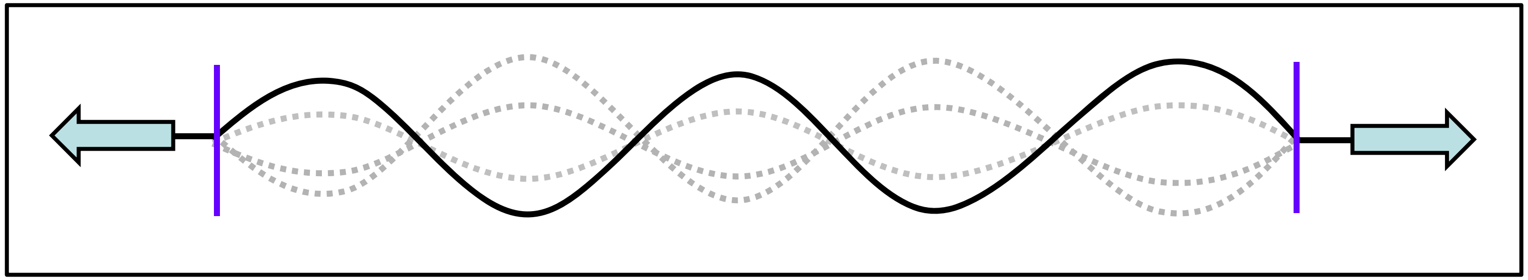
$$a(t) \equiv a_0$$

$$\begin{aligned}
E'(t) =& \int_{-L}^{L} a_0^2 \Re \left\{ \partial_x u(t,x) \overline{\partial_x \partial_t u(t,x)} \right\} dx \\
&+ \int_{-L}^{L} \Re \left\{ \partial_t^2 u(t,x) \overline{\partial_t u(t,x)} \right\} dx \\
=& \int_{-L}^{L} \Re \left\{ \left(-a_0^2 \partial_x^2 u(t,x) + \partial_t^2 u(t,x) \right) \overline{\partial_t u(t,x)} \right\} dx \\
=& 0
\end{aligned}$$

$\Longrightarrow$ $E(t) \equiv E(0)$ (Energy conservation law)

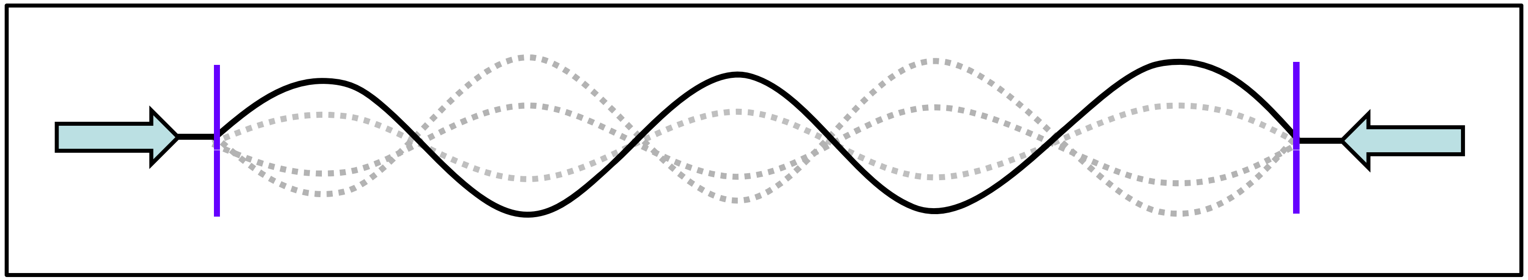
Increasing propagation speed

$$a'(t) > 0 \ \Rightarrow \ E'(t) = a'(t)a(t)\int_{-L}^{L} |\partial_x u(t,x)|^2 dx \geq 0$$



Decreasing propagation speed

$$a'(t) < 0 \ \Rightarrow \ E'(t) = a'(t)a(t)\int_{-L}^{L} |\partial_x u(t,x)|^2 dx \leq 0$$



The sign and the order of $a'(t)$ should be crucial for the asymptotic behavior or the energy

Oscillating propagation speed

$$E'(t) = a'(t)a(t)\int_{-L}^{L} |\partial_x u(t,x)|^2 dx \begin{cases} \leq \frac{2|a'(t)|}{a(t)} E(t) \\ \geq -\frac{2|a'(t)|}{a(t)} E(t) \end{cases}$$

$$\Longrightarrow \quad E(0)\, e^{-\eta(t)} \leq E(t) \leq E(0)\, e^{\eta(t)}$$

$$\eta(t) = \int_0^t \frac{2|a'(s)|}{a(s)} ds$$

$$|a'(t)| \leq C \;\Rightarrow\; \eta(t) \simeq 1 + t$$

$$|a'(t)| \leq C(1+t)^{-\beta},\ \beta \in (0,1) \;\Rightarrow\; \eta(t) \simeq (1+t)^{1-\beta}$$

$$|a'(t)| \leq C(1+t)^{-1} \;\Rightarrow\; \eta(t) \simeq \log(e+t)$$

$$|a'(t)| \leq C(1+t)^{-\beta},\ \beta > 1 \;\Rightarrow\; \eta(t) \simeq 1$$

These estimates are not taken into account any property of compensation of the oscillation of the coefficient.

If we can derive a benefit of the oscillation, then we may improve the energy estimate.

<u>Problem</u>

Find the conditions to $a(t)$ to conclude the estimate:

$$E(t) \simeq E(0) \ (\Leftrightarrow \ C_0 E(0) \leq E(t) \leq C_1 E(t))$$

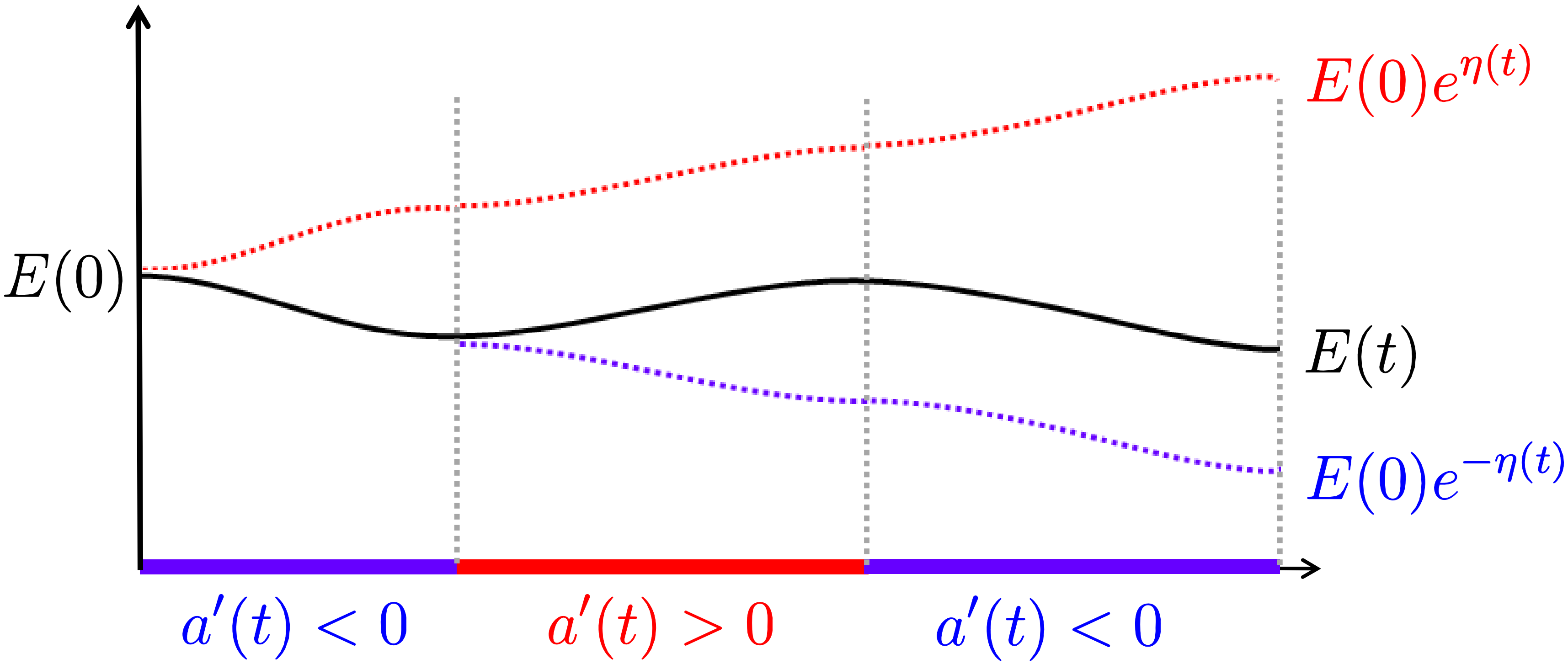
Generalized Energy Conservation (= GEC)

<u>Motivation</u>

Derive a compensation of the oscillation of the coefficient

Derive a smoothness of the coefficient

Compensation of the oscillation of the coefficient



How can we realize the compensation of the oscillation of the coefficient?

- Classical energy estimate with Gronwall's lemma is too rough
- (Almost) impossible to solve PDEs with variable coefficients

Smoothness of the coefficient

No (GEC) for Hölder continuous coefficient:
[Colombini; De Giorgi; Spagnolo, *A.S.N.S. Pisa.* (1979)]

(GEC) is valid for β=1 if $a(t) \in C^2$:
[Reissig; Smith, *Hokkaido M.J.* (2005)]

Theorem 0. $|a'(t)| \lesssim (1+t)^{-\beta}$, $\beta > 1 \Rightarrow$ (GEC)

$$(\partial_t^2 - a_0^2 \partial_x^2)\, u(t,x) = 0$$

What condition to $a(t)$ does conclude
a small perturbation in the case of $a(t) = a_0$?
(smallness of $\|a(t) - a_0\|_H$ in a norm space H)

$$(\partial_t^2 - a(t)^2 \partial_x^2)\, u(t,x) = 0$$

2. Main Theorem

(1) $$\begin{cases} (\partial_t^2 - a(t)^2 \partial_x^2)\, u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times [-L, L] \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in [-L, L] \\ u(t,-L) = u(t,L) = 0, \quad t \in \mathbb{R}_+ \end{cases}$$

(2) $$E(t) = \frac{1}{2} a(t)^2 \int_{-L}^{L} |\partial_x u(t,x)|^2 dx + \frac{1}{2} \int_{-L}^{L} |\partial_t u(t,x)|^2 dx$$

$$a(t) \in C^m(\mathbb{R}_+) \quad (m \geq 2)$$

(3) $$\left| \frac{d^k}{dt^k} a(t) \right| \leq C_k (1+t)^{-\beta k} \quad (k = 1, \cdots, m)$$

Main theorem. $\beta > \frac{1}{m} \Rightarrow$ (GEC).

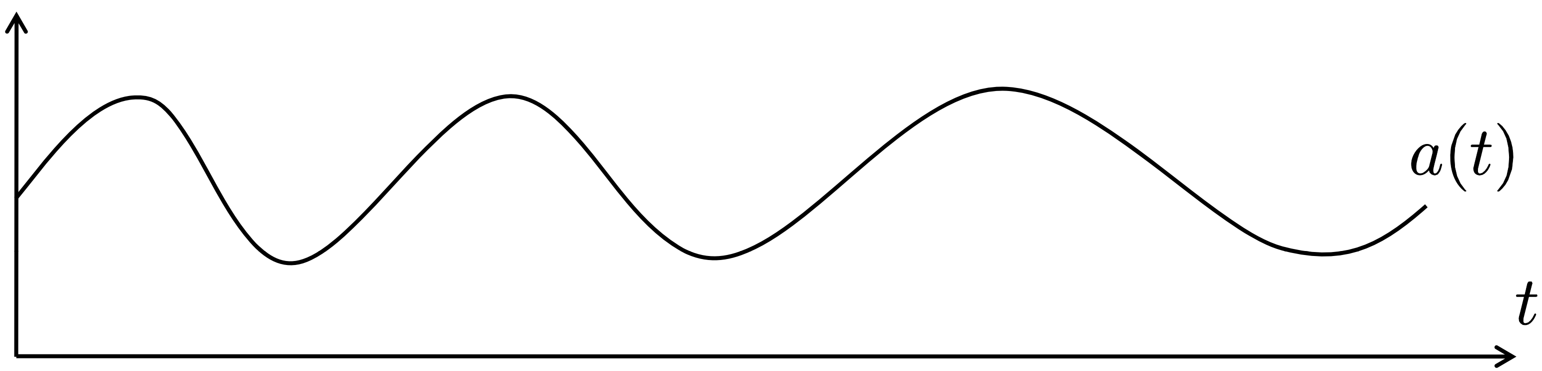
Main theorem. $\beta > \frac{1}{m} \Rightarrow \text{(GEC)}.$

Corollary. $m = \infty,\ \beta > 0 \Rightarrow \text{(GEC)}.$

Example. $a(t) = p\left((1+t)^{1-\beta}\right) \quad (0 < \beta < 1);$

$p(t) \in C^m(\mathbb{R})$, positive, 1-periodic;

$|a^{(k)}(t)| \leq C_k(1+t)^{-\beta k} \ (k = 1, \cdots, m)$



Remark. Corresponding result to the Cauchy problem:

$$\begin{cases} (\partial_t^2 - a(t)^2\partial_x^2)\, u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R} \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R} \end{cases}$$

with the energy

$$E(t) = \frac{1}{2}a(t)^2 \int_{\mathbb{R}} |\partial_x u(t,x)|^2 dx + \frac{1}{2}\int_{\mathbb{R}} |\partial_t u(t,x)|^2 dx$$

is considered in [H. *Math. Ann.* (2007)]

3. Sketch of the Proof

$$(\partial_t^2 - a(t)^2\partial_x^2)\, u(t,x) = 0 \text{ in } \mathbb{R}_+ \times \Omega \quad (\Omega = [-L, L])$$

$$u(t,x) = \sum_{k=1}^{\infty} v_k(t) w_k(x), \quad -\partial_x^2 u(t,x) = \sum_{k=1}^{\infty} \lambda_k^2 v_k(t) w_k(x)$$

$\{w_k(x)\}_{k=1}^{\infty}$: CONS in $L^2(\Omega)$

$\{v_k(t)\}_{k=1}^{\infty}$: Fourier coefficients

$\{\lambda_k^2\}_{k=1}^{\infty}$: eigenvalus of $-\partial_x^2$ with $u(t,-L) = u(t,L) = 0$

$$0 < \lambda_1 \le \lambda_2 \le \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty$$

$$\left(\frac{d^2}{dt^2} + a(t)^2\lambda_k^2\right) v_k(t) = 0 \quad (k = 1, 2, \cdots)$$

$$E(t) = \sum_{k=1}^{\infty} \mathcal{E}_k(t), \quad \mathcal{E}_k(t) = \frac{1}{2}\left(a(t)^2\lambda_k^2 |v_k(t)|^2 + |v_k'(t)|^2\right)$$

$$\left(\frac{d^2}{dt^2} + a(t)^2\lambda_k^2\right) v_k(t) = 0 \quad (k = 1, 2, \cdots)$$

$$E(t) = \sum_{k=1}^{\infty} \mathcal{E}_k(t), \quad \mathcal{E}_k(t) = \frac{1}{2}\left(a(t)^2\lambda_k^2|v_k(t)|^2 + |v_k'(t)|^2\right)$$

<u>Our goal</u>

Prove the following estimate uniformly with respect to k:

$$C_0\mathcal{E}_k(0) \leq \mathcal{E}_k(t) \leq C_1\mathcal{E}_k(0)$$

Remark. For any given $T > 0$ the following estimate is trivial:

$$C_0\mathcal{E}_k(T) \leq \mathcal{E}_k(t) \leq C_1\mathcal{E}_k(T) \quad (\forall t \geq T)$$

$$\left(\frac{d^2}{dt^2} + a(t)^2\lambda^2\right) v(t) = 0 \quad (\lambda = \lambda_k,\ v(t) = v_k(t))$$

$$V = \begin{pmatrix} ia\lambda v \\ v' \end{pmatrix}, \quad A = \begin{pmatrix} \frac{a'}{a} & ia\lambda \\ ia\lambda & 0 \end{pmatrix}$$

$$\left(\frac{d}{dt} - A\right) V = 0$$

$$V_1 = M_0 V, \quad M_0 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Phi_1 = \mathrm{diag}(M_0 A M_0^{-1}), \quad R_1 = M_0 A M_0^{-1} - \Phi_1$$

$$\left(\frac{d}{dt} - \Phi_1 - R_1\right) V_1 = 0$$

$$(4) \quad \left(\frac{d}{dt} - \Phi_1 - R_1\right) V_1 = 0$$

$$V_1 = \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix}, \quad \Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & \overline{r_1} \\ r_1 & 0 \end{pmatrix}$$

$$v_1 = v' + ia\lambda v, \quad \phi_1 = \frac{a'}{2a} + ia\lambda, \quad r_1 = -\frac{a'}{2a}$$

Remark. Reduction to (4) ⇔ Classical energy estimate

$$\frac{1}{2}|V_1|^2 = |v_1|^2 = a^2\lambda^2|v|^2 + |v'|^2 = 2\mathcal{E}(t) \quad (\mathcal{E}(t) = \mathcal{E}_k(t))$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}|V_1|^2\right) &= \Re\left(V_1, V_1'\right)_{\mathbb{C}^2} = \Re\left(V_1, \Phi_1 V_1\right)_{\mathbb{C}^2} + \Re\left(V_1, R_1 V_1\right)_{\mathbb{C}^2} \\ &= \phi_{1,\Re}|V_1|^2 + 2\Re\left\{r_1 v_1^2\right\} \end{aligned}$$

$$\frac{d}{dt}\left(\frac{1}{2}|V_1|^2\right) = \phi_{1,\Re}|V_1|^2 + 2\Re\left\{r_1 v_1^2\right\} = -r_1|V_1|^2 + 2\Re\left\{r_1 v_1^2\right\}$$

$$\begin{cases} \le \left(-r_1 + |r_1|\right)|V_1|^2 = \left(\frac{a'}{2a} + \left|\frac{a'}{2a}\right|\right)|V_1|^2 \\ \ge \left(-r_1 - |r_1|\right)|V_1|^2 = \left(\frac{a'}{2a} - \left|\frac{a'}{2a}\right|\right)|V_1|^2 \end{cases}$$

$$\Longleftrightarrow \quad \left(\frac{a'}{a} - \frac{|a'|}{a}\right)\mathcal{E}(t) \le \mathcal{E}'(t) \le \left(\frac{a'}{a} + \frac{|a'|}{a}\right)\mathcal{E}(t)$$

$$\Longleftrightarrow \quad -\frac{|a'|}{a}\frac{\mathcal{E}(t)}{a(t)} \le \left(\frac{\mathcal{E}(t)}{a(t)}\right)' \le \frac{|a'|}{a}\frac{\mathcal{E}(t)}{a(t)}$$

$$\left(\frac{\mathcal{E}(t)}{a_1} \le \frac{\mathcal{E}(t)}{a(t)} \le \frac{\mathcal{E}(t)}{a_0} \;\Leftrightarrow\; \frac{\mathcal{E}(t)}{a(t)} \simeq \mathcal{E}(t)\right)$$

$$\Longrightarrow \quad \mathcal{E}(t) \simeq \mathcal{E}(0) \;\; \text{if} \;\; |a'(t)| \le C(1+t)^{-\beta}, \; \beta > 1$$

$$\Longrightarrow \quad E(t) \simeq E(0) \;\; \text{(GEC)}$$

$$\left(\frac{d}{dt} - \Phi_1 - R_1\right) V_1 = 0, \quad \Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & \overline{r_1} \\ r_1 & 0 \end{pmatrix}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}|V_1|^2\right) &= \Re\left(V_1, \Phi_1 V_1\right)_{\mathbb{C}^2} + \Re\left(V_1, R_1 V_1\right)_{\mathbb{C}^2} \\ &= \phi_{1,\Re}|V_1|^2 + 2\Re\left\{r_1 v_1^2\right\} \begin{cases} \leq \left(-\phi_{1,\Re} + |r_1|\right)|V_1|^2 \\ \geq \left(-\phi_{1,\Re} - |r_1|\right)|V_1|^2 \end{cases} \end{aligned}$$

$\Im\{\Phi_1\}$: no effect to the energy estimates

$\Re\{\Phi_1\}$: describes an oscillation of the energy

R_1: error (compensation of the oscillation is not derived)

Diagonalization is essential!

Summary

$$\left(\frac{d}{dt} - \Phi - R\right) W = 0, \quad \phi_{\Re} = \Re\{\phi\}, \quad \phi_{\Im} = \Im\{\phi\}$$

$$\sup_{t \geq T} \left\{ \left| \int_T^t \phi_{\Re}\, ds \right| \right\} < \infty, \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & \overline{\phi} \end{pmatrix}, \quad R = \begin{pmatrix} 0 & \overline{r} \\ r & 0 \end{pmatrix}$$

$$\Longrightarrow \quad |W(t)| \begin{cases} \lesssim |W(T)| \exp\left(C \int_T^t |r(s)|\, ds \right) \\ \gtrsim |W(T)| \exp\left(-C \int_T^t |r(s)|\, ds \right) \end{cases}$$

$$\Longrightarrow \quad |W(t)| \simeq |W(T)| \quad \text{if} \quad \sup_{t \geq T} \left\{ \int_T^r |r(s)| ds \right\} < \infty$$

$$\left(\frac{d}{dt} - \Phi_1 - R_1\right) V_1 = 0, \quad \Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & \overline{r_1} \\ r_1 & 0 \end{pmatrix}$$

$$V_2 = M_1^{-1} V_1, \quad M_1 = \begin{pmatrix} 1 & \overline{\delta_1} \\ \delta_1 & 1 \end{pmatrix}, \quad \delta_1 = \frac{-i r_1}{2\phi_{1,\Im}}$$

$$|\delta_1| = \frac{\left|\frac{a'}{2a}\right|}{2a\lambda} \leq \frac{C_1(1+t)^{-\beta}}{4a_0^2\lambda} \leq \frac{1}{2} \quad \left(t \geq T_1 := \left(\frac{C_1}{2a_0^2\lambda_1}\right)^{\frac{1}{\beta}} - 1\right)$$

$$\left(\frac{d}{dt} - \Phi_2 - R_2\right) V_2 = 0, \quad \Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & \overline{r_2} \\ r_2 & 0 \end{pmatrix}$$

$$\phi_{2,\Re} = \frac{1}{2}\frac{d}{dt}\left(\log\left(\frac{a}{1-|\delta_1|^2}\right)\right), \quad \phi_{2,\Im} = a\lambda - \frac{2|\delta_1|^2}{1-|\delta_1|^2}$$

$$r_2 = \frac{r_1|\delta_1|^2 - \delta_1'}{1-|\delta_1|^2}$$

$$\left(\frac{d}{dt} - \Phi_2 - R_2\right) V_2 = 0, \quad \Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & \overline{r_2} \\ r_2 & 0 \end{pmatrix}$$

$$\phi_{2,\Re} = \frac{1}{2}\frac{d}{dt}\left(\log\left(\frac{a}{1-|\delta_1|^2}\right)\right), \quad r_2 = \frac{r_1|\delta_1|^2 - \delta_1'}{1-|\delta_1|^2}$$

$$\delta_1 = \frac{-ir_1}{2\phi_{1,\Im}} = \frac{ia'}{4a^2\lambda}, \quad r_1 = -\frac{a'}{2a}$$

$$\sup_{t>T_1}\left\{\left|\int_{T_1}^{t} \phi_{2,\Re}(s)\,ds\right|\right\} < \infty, \quad |r_2(t)| \le C(1+t)^{-2\beta} \quad (t \ge T_1)$$

$$|V_2(t)|^2 = |M_1(t)^{-1}V_1(t)|^2 \simeq |V_1(t)|^2 \simeq \mathcal{E}(t) \quad (t \ge T_1)$$

$$\Downarrow$$

$$\mathcal{E}(t) \simeq \mathcal{E}(T) \ \text{ if } \ \beta > \frac{1}{2} \quad (t \ge T_1)$$

$$\left(\frac{d}{dt} - \Phi_j - R_j\right) V_j = 0$$

$$V_j = \begin{pmatrix} v_j \\ \overline{v_j} \end{pmatrix}, \Phi_j = \begin{pmatrix} \phi_j & 0 \\ 0 & \overline{\phi_j} \end{pmatrix}, \; B_j = \begin{pmatrix} 0 & \overline{r_j} \\ r_j & 0 \end{pmatrix}$$

$$V_{j+1} = M_j^{-1} V_j, \quad M_j = \begin{pmatrix} 1 & \overline{\delta_j} \\ \delta_j & 1 \end{pmatrix}, \quad \delta_j = \frac{-i r_j}{2\phi_{j,\Im}}$$

$$\left(\frac{d}{dt} - \Phi_{j+1} - R_{j+1}\right) V_{j+1} = 0$$

$$\exists M_j \; (\forall t > T_j \gg 1)$$

$$\begin{cases} \phi_{j+1,\Re} = \dfrac{1}{2}\dfrac{d}{dt}\left(\log\left(\dfrac{a}{\prod_{k=1}^{j}\left(1-|\delta_k|^2\right)}\right)\right) \\ \phi_{j+1,\Im} = a\lambda + \displaystyle\sum_{k=1}^{j} \dfrac{-2|\delta_k|^2\phi_{k,\Im} + \Im\{\delta_k' \overline{\delta_k}\}}{1-|\delta_k|^2} \\ r_{j+1} = \dfrac{r_j|\delta_j|^2 - \delta_j'}{1-|\delta_j|^2} \quad (j = 0, \cdots, m-1) \end{cases}$$

$$\left(\frac{d}{dt} - \Phi_m - R_m\right) V_m = 0, \quad \Phi_m = \begin{pmatrix} \phi_m & 0 \\ 0 & \overline{\phi_m} \end{pmatrix}, \quad R_m = \begin{pmatrix} 0 & \overline{r_m} \\ r_m & 0 \end{pmatrix}$$

$$\phi_{m,\Re} = \frac{1}{2}\frac{d}{dt}\left(\log\left(\frac{a}{\prod_{k=1}^{m-1}(1-|\delta_k|^2)}\right)\right), \quad r_m = \frac{r_{m-1}|\delta_{m-1}|^2 - \delta'_{m-1}}{1-|\delta_{m-1}|^2}$$

$$\sup_{t>T_{m-1}}\left\{\left|\int_{T_{m-1}}^{t} \phi_{m,\Re}(s)\,ds\right|\right\} < \infty$$

$$|r_{m-1}(t)| \le C(1+t)^{-(m-1)\beta}$$

$$|r_m(t)| \le C(1+t)^{-m\beta} \quad (t \ge T_{m-1})$$

$$|V_m(t)|^2 = |M_{m-1}(t)^{-1}\cdots M_1^{-1}V_1(t)|^2 \simeq |V_1(t)|^2 \simeq \mathcal{E}(t) \quad (t \ge T_{m-1})$$

$$\mathcal{E}(t) \simeq \mathcal{E}(T_{m-1}) \quad \text{if} \quad \beta > \frac{1}{m} \quad (t \ge T_{m-1})$$

Summary

(i) Reduction to 2nd order ODEs of Fourier coefficients

$$\left(\frac{d^2}{dt^2}+a(t)^2\lambda_k^2\right)v_k(t)=0\ \ (k=1,2,\cdots)$$

(ii) Reduction to 1nd order ODE system

(iii) Diagonalization 1
(C^1 property and hyperbolicity ⇔ classical energy method)

$$\left(\frac{d}{dt}-\Phi_1-R_1\right)V_1=0$$

(iv) Diagonalization 2 (C^m property, large t)

$$M_j^{-1}\left(\frac{d}{dt}-\Phi_j-R_j\right)M_j=\frac{d}{dt}-\Phi_{j+1}-R_{j+1}$$

Smoother coefficient contributes better energy estimate

Thank you very much!