A remark on the asymptotic profile of solutions to the damped wave equation with variable coefficients

Yuta Wakasugi
Nagoya University, Japan
email: yuta.wakasugi@math.nagoya-u.ac.jp

Abstract In this talk, we consider the Cauchy problem to the nonlinear damped wave equation

$$\begin{cases} u_{tt} - (a(x)u_x)_x + b(t)u_t = N(u, u_x, u_t), & t > 0, x \in \mathbb{R}, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R} \end{cases}$$

and we study the asymptotic behavior of solutions as $t \to +\infty$. When $a(x) \equiv 1$, $b(t) = (1+t)^{-\beta}$ $(-1 < \beta < 1)$ and $N(u, u_x, u_t) = -|u|^{p-1}u$ $(p > 3)$, Nishihara [2] proved that the asymptotic profile of the solution with small initial data is given by the constant multiple of the heat kernel of the corresponding parabolic equation. In this talk, we extend the result of [2] to more general $a(x)$, $b(t)$ and $N(u, u_x, u_t)$. We show that if $a(x)$ has the positive limits $a_\pm$ as $x \to \pm\infty$, $b(t)$ behaves as $b(t) \sim (1+t)^{-\beta}$ with some $-1 < \beta < 1$, and the nonlinear term $N(u, u_x, u_t)$ can be regarded as a perturbation, then the asymptotic profile of the solution with small initial data is given by the constant multiple of the heat kernel of the corresponding parabolic equation. For the proof, following Gallay and Raugel [1], we use the scaling variables defined by

$$y = B(t)^{-1/2}x, \quad s = \log B(t) \quad \text{with } B(t) = \int_0^t \frac{d\tau}{b(\tau)} + 1$$

and we put $B(t)^{1/2}u(t, x) = v(s, y)$ and $b(t)B(t)^{3/2}u_t(t, x) = w(s, y)$. Then we have the first order system

$$\begin{cases} v_s - \dfrac{y}{2}v_y - \dfrac{1}{2}v = w, \\ \dfrac{e^{-s}}{b(t)^2}\left(w_s - \dfrac{y}{2}w_y - \dfrac{3}{2}w\right) - \dfrac{b'(t)}{b(t)^2}w + w = (a(e^{s/2}y)v_y)_y + \tilde{N}(t, v, w). \end{cases}$$

Then we show that the solution (v, w) can be expanded by eigenfunctions of the limiting operator

$$\tilde{\mathcal{L}}v = (\tilde{a}(y)v_y)_y + \frac{y}{2}v_y + \frac{1}{2}v,$$

where $\tilde{a}(y)$ is defined by $\tilde{a}(y) = a_-$ $(y < 0)$ and $\tilde{a}(y) = a_+$ $(y > 0)$.

Bibliography

[1] Th. Gallay and G. Raugel, *Scaling variables and asymptotic expansions in damped wave equations*, J. Differential Equations **150** (1998), 42–97.

[2] K. Nishihara, *Asymptotic profile of solutions for 1-D wave equation with time-dependent damping and absorbing semilinear term*, Asymptotic Analysis **71** (2011), 185–205.