Integral transform approach to the time-dependent partial differential equations

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Abstract In this talk we present an integral transform that allows to write solutions of the partial differential equation with variable coefficients via solutions of simpler equation.

Consider for the smooth function $f = f(x,t)$ the solution $w = w(x,t;b)$ to the problem

$$(1) \qquad w_{tt} - A(x,\partial_x)w = 0,\ \ w(x,0;b) = f(x,b),\ \ w_t(x,0;b) = 0,\ \ t \in [0,T_1],\ \ x \in \Omega \subseteq \mathbb{R}^n,$$

with the parameter $b \in [t_{in},T]$, $0 \le t_{in} < T \le \infty$, and with $0 < T_1 \le \infty$. Here Ω is a domain in $\mathbb{R}^n$, while $A(x,\partial_x) = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$ is the partial differential operator with smooth coefficients, $a_\alpha \in C^\infty(\Omega)$. We are going to present the integral operator

$$(2) \qquad \mathcal{K}[w](x,t) = \int_{t_{in}}^{t} db \int_0^{|\phi(t)-\phi(b)|} K(t;r,b)w(x,r;b)\,dr, \quad x \in \Omega,\ t \in [t_{in},T],$$

which maps the function $w = w(x,r;b)$ into the solution $u = u(x,t)$ of the equation

$$u_{tt} - a^2(t)A(x,\partial_x)u + M^2u = f, \qquad x \in \Omega\,,\ t \in [t_{in},T]\,,$$

where $a^2(t) = e^{\pm 2t}$, t^ℓ, $\ell \in \mathbb{C}$, $M \in \mathbb{C}$. In fact, the function $u = u(x,t)$ takes initial values

$$u(x,t_{in}) = 0,\ \ u_t(x,t_{in}) = 0, \quad x \in \Omega\,.$$

In (2), $\phi = \phi(t)$ is a distance function produced by $a = a(t)$, that is $\phi(t) = \int_{t_{in}}^t a(\tau)\,d\tau$. Moreover, we also introduce the corresponding operators, which generate solutions of the source-free equation and takes non-vanishing initial values.

We illustrate this approach by application to several model equations. In particular, we give applications to the generalized Tricomi equation, the Klein-Gordon and wave equations in the curved spacetimes such as de Sitter, Einstein-de Sitter, anti-de Sitter, Schwarzschild, and Schwarzschild-de Sitter spacetimes. The particular version of this transform was used in a series of papers [1, 2, 3, 4, 5] to investigate in a unified way several linear and semilinear equations. The results on the global existence of the small data solutions of the Cauchy problem for the semilinear Tricomi equation, the system of semilinear Klein-Gordon equations in the de Sitter spacetime, were established. The relations to the Higuchi bound and Huygens Principle were revealed as well.

Bibliography

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