NON CUTOFF BOLTZMANN COLLISION OPERATORS LOWER AND UPPER BOUNDS

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ABSTRACT. In this talk, we present some results on lower and upper bounds for Boltzmann collision operators, for non cutoff cross sections. The mains results presented were obtained through a series of works in collaboration with Y. Morimoto, S. Ukai C.-J. Xu and T. Yang.

1. GENERAL INTRODUCTION

The following Notes are a written version of a talk given during the conference dedicated to the 60*th* birthday of Prof. Yoshinori Morimoto. I have kept the informal way of the talk, and we refer to a complete set of results to the papers by the AMUXY group (Alexandre, Morimoto, Ukai, Xu, Yang).

General references on Boltzmann equation are detailed in the classical books of Cercignani or Chapman and Cowling [12, 13, 14]. For the precise case of non cutoff cross sections, we refer to the reviews by Alexandre [1] and the one by Villani [27]. Finally, we refer to the bibliography for the full set of details.

We consider the Boltzmann equation with *singular* kernels, i.e. collisional kernels which *do not* satisfy Grad's cutoff assumption, also called *non cutoff* kernels.

Let us recall that Boltzmann equation read as follows

$$\partial_t f(t,x,v) + v.\nabla_x f(t,x,v) = Q(f,f)(t,x,v), \tag{1}$$

where time $t \geq 0$, position $x \in \mathbb{R}^n$, velocity $v \in \mathbb{R}^n$, for $n \geq 2$. .

On the right hand side of (1),we have the Boltzmann collision operator, which here acts only w.r.t. velocity variable v:

$$Q(f,f)(v) = \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\sigma B(v-v_*,\sigma)(f'f'_* - ff_*). \tag{2}$$

For a given couple of pre-collisional velocities (v,v_*), the post collisional velocities are given in the $\sigma-$representation by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

In particular, we have the usual conservation of momentum and energy laws:

$$v' + v'_* = v + v_* \text{ and } |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

B is the collisional cross sections and given through the usual two points scattering process, and generally one has the following dependance

$$B(v-v_*,\sigma) = B(|v-v_*|, \frac{v-v_*}{|v-v_*|}\cdot\sigma) \text{ and } \cos\theta = \frac{v-v_*}{|v-v_*|}\cdot\sigma.$$

Roughly sepaking, the mathematical theory of Boltzmann equation is divided into two main research directions:

(1) *spatially homogeneous* equation, that is solutions independent of position variable x;
(2) *spatially inhomogeneous* case, that is full Boltzmann equation.

However, another mathematical (and physical) separation stands upon whether or not, for a fixed $z \in \mathbb{R}^n - \{0\}$, the function $\sigma \mapsto B(|z|,\sigma)$ is integrable on the unit sphere S^{n-1}.

At the exception of the hard sphere case, this function is never integrable, because of a non integrability behavior for small deviation angles. This explains why the usual cutoff assumption was introduced by Grad [21] in order to simplify the mathematical analysis. This is the main assumption used in most works up to the 1990's on Boltzmann equation, see [1, 27] and references therein for more details. Nevertheless, before the 1990's, we can cite the main works on singular kernels by Arkeryd, Pao and Ukai [10, 25, 26].

For example, a typical and classical physical example of non cutoff cross-sections is given by inverse power laws interactions, $\phi(r) = \frac{1}{r^{\alpha-1}}$, $\alpha > 2$. A good approximation is given by

$$B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma b(\cos\theta),\ \sin^{n-2}\theta b(\cos\theta) \sim_{\theta\to 0} K\theta^{-1-\nu}$$

where $\gamma = \frac{\alpha-5}{\alpha-1}$ and $\nu = \frac{2}{\alpha-1}$ if $n=3$.

Starting from the 1990's, Desvillettes [15, 16, 17] introduced some pioneering works which were the start-off for the non cutoff theory. In particular, he showed that singular kernels lead to very different effects compared to the non singular case, solutions to Boltzmann spatially homogeneous equation enjoy *regularization properties!*

Let us mention that another well known equation which is closely related to Boltzmann equation is the Landau equation

$$Q^L(f,f)(v) = \nabla_v.(\int_{\mathbb{R}^n} dv_*\mathbf{a}(v-v_*)[f_*(\nabla f) - f(\nabla f)_*]dv_*,$$

where $\mathbf{a}(z) = |z|^2\Phi(z)\Pi_{z^\perp}$, $\Pi_{z^\perp}$ being the orthogonal projection onto $z^\perp$ that is

$$(\Pi_{z^\perp})_{i,j} = \delta_{i,j} - \frac{z_iz_j}{|z|^2}.$$

Function Φ comes from $B(v-v_*,\cos\theta) = \Phi(v-v_*)b(\cos\theta)$.

Landau equation is extremely important in plasman Physics, but it is also useful for mathematical insights on Boltzmann equation, see for example the works [18, 23] and other references mentioned in the two reviews mentioned earlier. Moreover, as explained for example therein, other representations of the collisional operator are also available. For instance, the one above is called the σ*-representation*. Another well known representation is the ω*-representation*

$$Q(f,f)(v) = \int_{\mathbb{R}^n} dv_* \int_{S^{n-1}} d\omega[f'f'_* - ff_*]2^{n-2}\sin^{n-2}(\frac{\theta}{2})B(|v-v_*|,\cos\theta),$$

$$v' = v + (v_*-v).\omega\ \omega \text{ and } v'_* = v_* - (v_*-v).\omega\ \omega.$$

Finally, a third representation is the *Carleman* representation: letting $E_{v,v-v'}$ be the hyperplane going through v and orthogonal to $v-v'$,

$$\left\{ \begin{array}{c} Q(f,f)(v) = \\ \displaystyle\int_{\mathbb{R}^n} dv' \int_{E_{v,v-v'}} \frac{dv'_*}{|\ v-v'\ |^{n-1}} B(2v-v'-v'_*, \frac{v'-v'_*}{|\ v'-v'_*\ |})[.] \\ \\ \text{with } [.] \equiv [f(v'_*)f(v') - f(v)f(v'+v'_*-v)]. \end{array} \right.$$

These different representations have all proven useful for the study of functional properties of the Boltzmann collision operator.

2. Weak formulation

Before explaining refined estimates on Boltzmann operator, let us first of all explain how a weak sense could be given. Again we refer to [1, 27] for details and further comments. Assume for example that $n=3$, and

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta),\ \int_0^\pi \theta^2 b(\cos\theta)d\theta < +\infty.$$

For a suitable test function $\phi = \phi(v)$, classical arguments give

$$\int_{\mathbb{R}^3} dv Q(f,f)(v)\phi(v) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dv dv_* B(v-v_*,\sigma) f f_* \{\phi' + \phi'_* - \phi - \phi_*\}.$$

Then, using Taylor's formula, we note that

$$\phi' - \phi = (v'-v).\nabla\phi(v) + (v'-v)\otimes(v'-v) : \int_0^1 (1-t)D^2\phi(v+t(v'-v))dt$$

and

$$\phi'_* - \phi_* = (v'_*-v_*).\nabla\phi(v_*) + (v'_*-v_*)\otimes(v'_*-v_*) : \int_0^1 (1-t)D^2\phi(v_*+t(v'_*-v_*))dt.$$

It follows that

$$\phi' + \phi'_* - \phi - \phi_* = O(|v-v_*|^2\theta\wedge 1).$$

The above computations imply that for small singularities $0<\nu<1$ and for $\gamma \geq -2$, $Q(f,f)$ can be defined by duality.

The next question is: what can we do for higher singularities, $1 \leq \nu < 2$? We need to introduce more precised assumptions and notations.

Assume that $B(v-v_*,\sigma)$ is supported in the set $(0 \leq \theta \leq \pi/2)$ and

$$B(v-v_*,\sigma) = |v-v_*|^\gamma b(\cos\theta), \qquad \sin^{n-2}\theta\, b(\cos\theta) \sim K\theta^{-1-\nu},$$

$\nu > 0$, $K > 0$, with

$$\gamma \geq -n, \qquad 0 \leq \nu < 2, \qquad \gamma + \nu < 2.$$

Introduce the following physical quantity, i.e. the momentum transfer:

$$\left\{ \begin{array}{l} \mathscr{M}(|v-v_*|) \equiv \displaystyle\int_{S^{n-1}} B(v-v_*,\sigma)(1-k\cdot\sigma)\,d\sigma \\ = |S^{n-2}| \displaystyle\int_0^{\frac{\pi}{2}} B(|v-v_*|,\cos\theta)(1-\cos\theta)\,\sin^{n-2}\theta\,d\theta, \end{array} \right.$$

where $k = \dfrac{v-v_*}{|v-v_*|}$ and $\cos\theta = k\cdot\sigma$.

For $0 \leq \alpha \leq 2$, let

$$\mathscr{M}^{\alpha}(|z|) = \int_{S^{n-1}} B(z,\sigma)\,(1-k\cdot\sigma)^{\frac{\alpha}{2}}\,d\sigma, \qquad k = \frac{z}{|z|}.$$

In contrast to the previous weak form, we use this time the following one (which might seems weaker)

$$\int_{\mathbb{R}^n} Q(f,f)\phi(v)dv = \int_{\mathbb{R}^{2n}} dv\,dv_*\,ff_*\left[\int_{S^{N-1}} B(v-v_*,\sigma)(\varphi'-\varphi)\,d\sigma\right],$$

which suggest the study, for given v_*, of the linear operator

$$\mathscr{T} : \varphi \longmapsto \int_{S^{n-1}} B(v-v_*,\sigma)(\varphi'-\varphi)\,d\sigma.$$

The following first result is given by

Proposition 1. [9] *For all $\varphi \in W^{2,\infty}(\mathbb{R}^n_v)$,*

$$|\mathscr{T}\varphi(v)| \leq \frac{1}{2}\|\varphi\|_{W^{2,\infty}}|v-v_*|\left(1+\frac{|v-v_*|}{2}\right)\mathscr{M}(|v-v_*|).$$

Moreover, for all $\alpha \in [0,2]$ and $\varphi \in W^{2,\infty}(\mathbb{R}^n_v)$,

$$|\mathscr{T}\varphi(v)| \leq 2\|\varphi\|_{W^{2,\infty}}(1+|v-v_*|)^{\alpha}\,\mathscr{M}_{\alpha}(|v-v_*|),$$

In particular, if B satisfies the hard potential case (or even with $\gamma \geq -2$), and if f satisfies the usual a priori entropic bounds, then, for all $R>0$, $Q(f,f)$ defined by duality as above belongs to $L^{\infty}([0,T];W^{-2,1}(B_R(v))))$, where $B_R(v)$ denotes $\{v \in \mathbb{R}^n, |v| \leq R\}$.

One of the main idea in the proof is to use symmetry for the first order term coming from Taylor's formula, so that it also cancels high singularities. This result enables to take care of the case $\gamma > -2$, but for $\gamma > -3$, one needs to make use of the entropy dissipation rate.

3. Pseudo-differential formulations ($\omega-$representation)

We have mentioned early the results of Desvillettes showing regularization effects. We want to show here that there is a (formal) way to see these effects (though not a proof). We start from the following formula

$$Q(f,g)(v) \equiv \int_{\mathbb{R}^3_{v_1}}\int_{S^2_{\omega}} dv_1 d\omega\{f(v')g(v_1') - f(v)g(v_1)\}\tilde{B}(\mid v-v_1\mid,\mid(\frac{v-v_1}{\mid v-v_1\mid},\omega)\mid),$$

where here the velocity dimension is 3 and we make use of the ω *representation*, with:

$$\tilde{B}(\mid v-v_1\mid,\mid(\frac{v-v_1}{\mid v-v_1\mid},\omega)\mid) \equiv \mid v-v_1\mid^{\gamma}\frac{1}{\mid(\frac{v-v_1}{|v-v_1|},\omega)\mid^{\nu}},$$

$$\gamma = \gamma(s) = \frac{s-5}{s-1}, \nu = \nu(s) = \frac{s+1}{s-1},$$

First one can show

Proposition 2. [1] *The non linear Boltzmann operator Q writes as*

$$Q(f,g)(v) = \int_{\mathbb{R}^3_h}\frac{2dh}{\mid h\mid^{\nu+2}}\int_{E_{0,h}}\{f(v-h)g(\alpha+v) - f(v)g(\alpha+v-h)\}\times$$
$$\times\{\mid\alpha\mid^2+\mid h\mid^2\}^{\frac{\gamma+\nu}{2}}d\alpha,$$

where $E_{0,h}$ denotes the plane containing 0 *and orthogonal to* $h \in \mathbb{R}^3$.

One has also the following formula

$$Q(f,g)(v) \equiv \int_{\mathbb{R}^3_{v_1}} \int_{S^2_\omega} \{f(v')g(v'_1) - f(v)g(v_1)\} \mid v - v_1 \mid^{\gamma+\nu} \frac{dv_1 d\omega}{\mid v' - v \mid^{\nu}}.$$

Then, write

$$Q(f,g)(v) = Q_1(f,g)(v) + Q_2(f,g)(v),$$

where

$$Q_1(f,g)(v) \equiv \int_{\mathbb{R}^3_{v_1}} \int_{S^2_\omega} \{f(v')g(v'_1) - f(v)g(v_1)\} \mid v_1 - v' \mid^{\gamma+\nu} \frac{dv_1 d\omega}{\mid v' - v \mid^{\nu}}.$$

It follows that

$$Q_1(f,g) = Q_{1,1}(f,g) + Q_{1,2}(f,g),$$

where

$$Q_{1,1}(f,g)(v) = \int_{\mathbb{R}^3_h} \frac{2dh}{\mid h \mid^{\nu+2}} \int_{E_{0,h}} \{f(v-h) - f(v)\} g(\alpha + v) \mid \alpha \mid^{\gamma+\nu} d\alpha.$$

The important point is given by the next result

Proposition 3. [1] *There exists a fixed constant C'_s depending only on s (or ν), such that (with the notations of pdo theory)*

$$Q_{1,1}(f,g)(v) = -C'_s \int_{\mathbb{R}^3_\alpha} d\alpha g(\alpha + v) \mid \alpha \mid^{\gamma+\nu} \mid S(\alpha).D \mid^{\nu-1} (f)(v),$$

where $S(\alpha)$ denotes the projection operator over the hyperplane $E_{0,\alpha}$.

At least formally, one can deduce a partial conclusion: $Q_{1,1}$ is some type of “negative” pseudo-differential operator with strictly positive order.

4. The cancellation Lemma

What have we shown? One part of Boltzmann operator behaves as a speudodifferential ”elliptic” operator. Though we did not give any functional properties on it, let us first consider the remaining part. To be more precise, we go back to the $\sigma-$*representation*, and write:

$$Q(f,f) = \int_{\mathbb{R}^n}\int_{S^{n-1}} dv_* d\sigma B(v - v_*, \sigma) f'_*[f' - f] + f \int_{\mathbb{R}^n}\int_{S^{n-1}} dv_* d\sigma B(v - v_*, \sigma)(f'_* - f_*),.$$

The second term is exactly dealt with the following result

Proposition 4. *(Cancellation Lemma* [3]*) For a.a. $v \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n \times S^{n-1}} dv_* \, d\sigma \, B(v - v_*, \sigma)(f'_* - f_*) = f *_v S,$$

where

$$S(|z|) = |S^{n-2}| \int_0^{\frac{\pi}{2}} d\theta \, \sin^{n-2}\theta \left[\frac{1}{\cos^n(\theta/2)} B\left(\frac{|z|}{\cos(\theta/2)}, \cos\theta\right) - B(|z|, \cos\theta). \right].$$

One important idea is the introduction of an important change of variables.

5. Coercivity results

We have seen:

(1) one part of the Boltzmann collision operator behaves as a negative fractional Laplacian type operator;
(2) the other part of this operator behaves exactly as a convolution type operator.

It appears that the study of the first term is difficult. Therefore, having in mind the regularization properties as shown by Desvillettes, we ask the following more simpler Question: can we get some kind of coercivity ? could we use the entropy dissipation estimate ?

The following is the final result taken from Alexandre., Desvillettes, Villani and Wennberg [3]. In the linearized case, these estimates have been improved by Alexandre, Morimoto, Ukai, Xu, Yang [5, 6, 7]. Moreover, we refer also to the optimal estimates proven by Gressman and Strain [19, 20].

Explaining only the result from [3], , let us assume that $n \geq 2$, the kinetic cross-section $\Phi(|z|) : \mathbb{R}^n \to \mathbb{R}^+$ is continuous and strictly positive for $z \neq 0$, and

$$\sin^{n-2}\theta\, b(\cos\theta) \sim \frac{K}{\theta^{1+\nu}} \qquad \text{as} \quad \theta \to 0, \qquad\qquad \nu > 0.$$

Let us introduce the generalized linear (w.r.t. f) entropy dissipation functional, for a given positive function g

$$D(g,f) = -\int_{\mathbb{R}^n} Q(g,f)\, \log f.$$

Proposition 5. [3] *There exists a constant $C_{g,R}$, depending only on b, $\|g\|_{L^1_1}$, $\|g\|_{L\log L}$, R, and on Φ, such that*

$$\|\sqrt{f}\|^2_{H^{\nu/2}(|v|<R)} \leq C_{g,R}\left[D(g,f) + \|g\|_{L^1_2}\|f\|_{L^1_2}\right].$$

We just mention the use of an important formula (see the Appendix of [3] for generalization) due to Bobylev [11] for the Fourier transform

$$\widehat{Q(g,f)}(\xi) = \int_{S^{n-1}} b(\sigma.\frac{\xi}{|\xi|})[\hat{g}(\xi^-)\hat{f}(\xi^+) - \hat{g}(0)\hat{f}(\xi)]d\sigma,$$

with $\xi^{\mp} = \frac{1}{2}[\xi \mp |\xi|\sigma]$.

6. Upper Estimates for regularized cases

These estimates were first proven in the regularized case, which means that the kinetic factor $|v-v_*|^\gamma$ which appears in the cross section is replaced by its smoothed version $< v-v_* >^\gamma$ (which includes in both cases the maxwellian potential case $\gamma = 0$), see the initial works in Lp Besov spaces in [2].

Then, more simple proofs in the usual (L^2) Sobolev space by Huo, Morimoto, Ukai, Xu, Yang [22], see also Alexandre., Morimoto, Ukai, Xu, Yang [4].

For example, one has

Proposition 6. [4] *Let $0 < s < 1$ and $\gamma \in \mathbb{R}$. Then for any m, $\alpha \in \mathbb{R}$,*

$$\|Q(f,g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \lesssim \|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)}$$

for all $f \in L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}^3_v)$ and $g \in H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)$.

Proposition 7. [4] *(Weighted Upper Estimates)* (1) *When* $0 < s < 1/2$*, we have*

$$\|Q(f,g)\|_{H^m_l(\mathbb{R}^3_v)} \lesssim \|f\|_{L^1_{\max\{l+\gamma^+,(\gamma+2s)^+\}}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{l+(2s+\gamma)^+}(\mathbb{R}^3_v)},$$

provided that $m \le 0$ *and* $0 \le m+2s$. (2) *When* $1/2 < s < 1$*, we have*

$$\|Q(f,g)\|_{H^m_l(\mathbb{R}^3_v)} \lesssim \|f\|_{L^1_{\max\{l+2s-1+\gamma^+,(2s+\gamma)^+\}}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{l+\max\{2s-1+\gamma^+,(2s+\gamma)^+\}}(\mathbb{R}^3_v)},$$

provided that $-1 < m \le 0$. (3) *When* $s = 1/2$*, we have the same form of estimate as above with* $2s-1$ *replaced by any small* $\kappa > 0$.

Finally, we wish to mention at this point the following result (lower bound) obtained earlier by (Alexandre, Morimoto, Ukai, Xu, Yang)

Proposition 8. *(Precised coercivity in the regularized case) Assume that* $\gamma \in \mathbb{R}$, $0 < s < 1$. *Let* $g \in L^1_{\max\{\gamma^+,2-\gamma^+\}} \cap L\log L(\mathbb{R}^3_v)$, $g \ge 0, \not\equiv 0$. *Then there exists a constant* $C_g > 0$ *depending only on* $B(v-v_*,\theta)$, $\|g\|_{L^1_{\max\{\gamma^+,2-\gamma^+\}}}$ *and* $\|g\|_{L\log L}$, *and* $C > 0$ *depending on* $B(v-v_*,\theta)$ *such that for any smooth function* $f \in H^1_{\gamma/2}(\mathbb{R}^3_v) \cap L^2_{\gamma^+/2}(\mathbb{R}^3_v)$, *we have*

$$-\Big(Q(g,f),\, f\Big)_{L^2(\mathbb{R}^3_v)} \ge C_g \|W_{\gamma/2} f\|^2_{H^s(\mathbb{R}^3_v)} - C\|g\|_{L^1_{\max\{\gamma^+,2-\gamma^+\}}(\mathbb{R}^3_v)} \|f\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^3_v)} .$$

7. Linearized framework in the Maxwellian Case

We shall end by mentioning some other estimates in the linearized framework, and for the maxwellian molecules case, see the works of Alexandre., Morimoto, Ukai, Xu, Yang [5]. That is, we assume

$$B(|v-v_*|,\cos\theta) = b(\cos\theta), \ \ \cos\theta = \Big\langle \frac{v-v_*}{|v-v_*|}\,,\sigma \Big\rangle,\ 0 \le \theta \le \frac{\pi}{2},$$

and

$$b(\cos\theta) \approx K\theta^{-2-2s}, \ \ \theta \to 0^+,$$

with $0 < s < \frac{1}{2}$.

Then, we linearize Boltzmann equation around a normalized Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Denote

$$\Gamma(g,h) = \mu^{-1/2} Q(\sqrt{\mu}\,g,\, \sqrt{\mu}\,h).$$

Then the linearized Boltzmann operator takes the form

$$\mathscr{L} g = \mathscr{L}_1 g + \mathscr{L}_2 g = -\Gamma(\sqrt{\mu}\,, g) - \Gamma(g,\, \sqrt{\mu}\,).$$

And the original problem is now reduced to the Cauchy problem for the perturbation g

$$\begin{cases} g_t + v\cdot\nabla_x g + \mathscr{L} g = \Gamma(g,g),\ t>0; \\ g|_{t=0} = g_0. \end{cases}$$

This problem is considered in [5] in the following weighted Sobolev spaces. For k, $\ell \in \mathbb{R}$, set

$$H^k_\ell(\mathbb{R}^6_{x,v}) = \Big\{ f \in \mathscr{S}'(\mathbb{R}^6_{x,v})\,;\, W^\ell f \in H^k(\mathbb{R}^6_{x,v}) \Big\}\,,$$

where $\mathbb{R}^6_{x,v} = \mathbb{R}^3_x \times \mathbb{R}^3_v$ and $W^\ell(v) = \langle v \rangle^\ell = (1+|v|^2)^{\ell/2}$ is the weight with respect to the velocity variable $v \in \mathbb{R}^3_v$. We have shown in [5] the global existence of solution in this class, under a smallness assumption on the initial data.

The main tools are of course to establish sharp estimates but one important idea was the introduction of a new coercive norm which controls the nonlinear perturbative operator.

Let us recall the linearized operator is given by

$$\Gamma(f,g)(v) = \mu^{-1/2} \iint b(\cos\theta)\Big(\sqrt{\mu'_*} f'_* \sqrt{\mu'} g' - \sqrt{\mu_*} f_* \sqrt{\mu}\, g\Big) dv_* d\sigma$$
$$= \iint b(\cos\theta) \sqrt{\mu_*}\Big(f'_* g' - f_* g\Big) dv_* d\sigma.$$

$\Big(\mathscr{L} g, g\Big)_{L^2(\mathbb{R}^3_v)} = 0$ if and only if $\mathbf{P} g = g$ where

$$\mathbf{P} g = \Big(a + b\cdot v + c|v|^2\Big)\sqrt{\mu},$$

with $a, c \in \mathbb{R}, b \in \mathbb{R}^3$. Here, $\mathbf{P}$ is the L^2-orthogonal projection onto the null space

$$\mathscr{N} = \mathrm{Span}\left\{ \sqrt{\mu}\,, v_1\sqrt{\mu}\,, v_2\sqrt{\mu}\,, v_3\sqrt{\mu}\,, |v|^2\sqrt{\mu}\ \right\}.$$

The following important result was proven by Mouhot and Strain

Proposition 9. [24] *There exists a constant $C > 0$ such that*

$$\Big(\mathscr{L} g, g\Big)_{L^2(\mathbb{R}^3_v)} \geq C \,\|(\mathbf{I} - \mathbf{P}) g\|^2_{L^2_s(\mathbb{R}^3_v)}\,.$$

Then, we have introduced in [5] the following non-isotropic norm associated with the cross-section $b(\cos\theta)$:

$$|||g|||^2 = \iiint b(\cos\theta)\mu_*\left(g' - g\right)^2 + \iiint b(\cos\theta) g_*^2 \big(\sqrt{\mu'}\ - \sqrt{\mu}\big)^2,$$

One has some upper bound of this non-isotropic norm by some weighted Sobolev norm:

Proposition 10. [5] *There exists $C > 0$ such that*

$$|||g|||^2 \leq C ||g||^2_{H^s_s} \tag{1}$$

for any $g \in H^s_s(\mathbb{R}^3_v)$.

Up to the kernel of $\mathscr{L}$, the equivalence between the non-isotropic norm and the Dirichlet form of $\mathscr{L}$ holds true as explained by

Proposition 11. [5] *For $g \in \mathscr{N}^\perp$, we have*

$$\Big(\mathscr{L} g, g\Big)_{L^2(\mathbb{R}^3_v)} \sim |||g|||^2.$$

Furthermore, the non-isotropic norm controls the Sobolev norm of both derivative and weight of order s:

Proposition 12. [5] *There exists $C > 0$ such that*

$$|||g|||^2 \geq C\big(||g||^2_{H^s} + ||g||^2_{L^2_s}\big). \tag{2}$$

In order to apply the energy method initiated by Y. Guo on one hand and T.-L. Liu, T. Yang and S.-H. Yu on the other hand, one has finally the nice result

Proposition 13. [5] *There exists $C > 0$ such that*

$$\left|\Big(\Gamma(f,g), h\Big)_{L^2(\mathbb{R}^3)}\right| \leq C\Big(||f||_{L^2_s}\, |||g||| + ||g||_{L^2_s}\, |||f|||\Big)\ |||h|||\,.$$

Finally we wish to mention some extremely important extensions:

• General singular kernels in series of works by AMUXY [7, 8] for perturbative solutions.

• See also related works by Gressman, Strain [19].

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