REGULARITY OF SOLUTIONS FOR A CLASS OF DEGENERATE PARTIAL DIFFERENTIAL EQUATIONS†

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Dedicated to the 60^{th} *birthday of Professor Yoshinori Morimoto*

1. Introduction

What happened in the case of spatially inhomogeneous problems?

Consider following spatially inhomogeneous Boltzmann equation:

$$\partial_t f + v\cdot\partial_x f = Q(f,\ f),\quad t\in\mathbb{R}_+,\ (x,v)\in\mathbb{R}^n\times\mathbb{R}^n.$$

$\partial_t + v\cdot\partial_x$: transport operator

$Q(f,\ f)$: Boltzmann's collision operator given by

$$Q(f,f)(v) = \int_{\mathbb{R}^n}\int_{\mathbb{S}^{n-1}} B(v-v_*,\tau)\big\{f(v'_*)f(v') - f(v_*)f(v)\big\}d\tau dv_*,$$

where $\mathbb{S}^{n-1}$ is the $(n-1)-$dimensional unit sphere and $\tau\in\mathbb{S}^{n-1}$, and $B(v-v_*,\tau)$ is called the Boltzmann's collision kernel.

$B(v-v_*,\tau)$ is a non-negative function depending only on $|v-v_*|$ and $\left\langle \frac{v-v_*}{|v-v_*|},\ \tau\right\rangle$, Thus

$$B(v-v_*,\tau) = B(|v-v_*|,\cos\theta),\quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|},\ \tau\right\rangle.$$

$\theta\in[0,\ \pi/2]$: deviation angle

In particular,

$$B(v-v_*,\tau) = \Psi(|v-v_*|)b(\cos\theta).$$

Two classical examples:

- Hard sphere collisions. In this case,
$$B(v-v_*,\tau) = B(|v-v_*|,\cos\theta) = |v-v_*|\sin\theta.$$
For fixed $|v-v_*|$, the function $\tau :\longrightarrow B(|v-v_*|,\ \tau)$ is integrable in $\mathbb{S}^{n-1}$.
- Grazing collisions. Above function is not integrable on $\mathbb{S}^{n-1}$, typically with high singularity near $\theta = 0$.

A classical example is given by inverse power laws, that is, any two particles, apart from a distance r, interact on each other by a force $\frac{1}{r^s}$, $s>2$. In this case

$$B(|v-v_*|,\cos\theta) = |v-v_*|^\gamma\, b(\cos\theta),\qquad \sin^{n-2}\theta\, b(\cos\theta)\sim_{\theta\to 0}\frac{K}{\theta^{1+\nu}},$$

where $K>0$, $\gamma = \frac{s-5}{s-1}$, and $\nu = 2/(s-1)$.

This singularity of the deviation angle θ causes additional difficulties in the mathematical treatment of Boltzmann equation.

When $\theta\sim 0$, then

$$b(\cos\theta)\sim\theta^{-n+1-\nu},$$

†Joint with Weixi Li and Chaojiang Xu

the collision operator behaves essentially as a fractional power of the Laplacian:

$$Q(f,\, f) = -C_f(-\triangle_v)^\sigma f + \text{high order regular terms},$$

where $\sigma \stackrel{\text{def}}{=} \frac{\nu}{2} \in (0,1)$.

Related to the study of regularity for the solutions of the spatially inhomogeneous Boltzmann equation, we consider following linear model:

$$\mathcal{P}f = \partial_t f + v\cdot\partial_x f + a(t,x,v)(-\triangle_v)^\sigma f = g, \tag{1.1}$$

where $0<\sigma<1$ and coefficient $a(t,x,v)$ is a strictly positive function.

In case of $\sigma = 1$, we get another model, i.e. Fokker-Planck equation

$$\mathcal{L}f = \partial_t f + v\cdot\partial_x f - a(t,x,v)\triangle_v f = g, \tag{1.2}$$

which is actually related to the following spatially inhomogeneous Landau equation:

$$\begin{cases} \partial_t f + v\cdot\partial_x f = \nabla_v\cdot\left\{\int_{\mathbb{R}^3} a(v-v_*)[f(v_*)\nabla_v f(v) - f(v)\nabla_v f(v_*)]dv_*\right\}, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.3}$$

where $x,\ v\in\mathbb{R}^3,\ t\ge 0$ and $a=(a_{ij})$ is a nonnegative symmetric matrix given by $a_{ij}(v) = (\delta_{ij} - \frac{v_iv_j}{|v|^2})|v|^{\gamma+2},\ \gamma\in[0,1]$.

Set

$$\begin{aligned} \bar{a}_{ij}(t,x,v) &= (a_{ij}*f)(t,x,v) = \int_{\mathbb{R}^3} a_{ij}(v-v_*)f(t,x,v_*)dv_*; \\ \bar{c} &= c*f, \qquad c = \sum_{i,j=1}^{3}\partial_{v_iv_j}a_{ij} = -2(\gamma+3)|v|^\gamma. \end{aligned}$$

Then the Cauchy problem (1.3) can be rewritten as

$$\begin{cases} \partial_t f + v\cdot\partial_x f = \displaystyle\sum_{i,j=1}^{3}\bar{a}_{ij}\partial_{v_iv_j}f - \bar{c}f, \\ f(0,x,v) = f_0(x,v), \end{cases} \tag{1.4}$$

Here we shall consider such kind of kinetic equations in which the diffusion coefficient is nonlinear function of velocity variable v. Namely, we shall consider following operator in $\mathbb{R}^{2n+1}$:

$$\begin{aligned} \mathcal{P} &= \partial_t + v\cdot\partial_x + a(t,x,v)(-\triangle_v)^\sigma, \\ \mathcal{L} &= \partial_t + v\cdot\partial_x - a(t,x,v)\triangle_v, \end{aligned} \tag{1.5}$$

where $\triangle_v$ is Laplace operator of velocity variables v, $a(t,x,v)$ is a strictly positive function in $\mathbb{R}^{2n+1}$.

Motivation to study the operators $\mathcal{P}$ and $\mathcal{L}$

- Study the smoothness effect of linearized operators for spatially inhomogenous Landau equations or Boltzmann equations without angular cutoff.
- Other physical backgrounds (cf. Helffer-Nier, *Lecture Notes in Math.*, **1862**, *Springer-Verlag, Berlin, 2005*).

Well-known results:

- C^∞ regularity has been obtained by Alexandre-Ukai-Morimoto-Xu-Yang (2007) for linear spatially inhomogeneous Boltzmann equation without angular cutoff (by using the uncertainty principle and micro-local analysis);
- (1.5) satisfies the Hörmander's condition$\Rightarrow$ (1.5) is C^∞-Hypoelliptic;

- By Derridj-Zuily (1973)$\Rightarrow$ (1.5) is G^s-Hypoelliptic for $s > 6$.
  Remark: *In general, $s > 6$ is not optimal, the best possible would be* $s \geq 3$ (cf. *L. P. Rothschild-E. M. Stein, Hypoelliptic differential operators and nilpotent groups, Acta. Math,* **137** *(1977), 248-315*).

Results of This Talk:

- Gevrey regularity for the weak solution of linear and semi-linear kinetic equation given in (1.1).
- Gevrey regularity for the weak solution of linear and semi-linear Fokker-Planck equation given in (1.2).

Main Results:

Theorem 1.1. *For any $s \geq 3$, if the positive coefficient $a(t,x,v)$ is in $G^s(\mathbb{R}^{2n+1})$, then the operator $\mathcal{L}$ given by (1.5) is G^s-hypoelliptic in $\mathbb{R}^{2n+1}$.*

(JDE, 246 (2009),320-339)

Theorem 1.2. *Let $0 < \sigma < 1$ and $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\right\}$. For any $s \geq \frac{2}{\delta}$, if the positive coefficient $a(t,x,v)$ is in $G^s(\mathbb{R}^{2n+1})$, then the operator $\mathcal{P}$ given by (1.1) is G^s hypoelliptic in $\mathbb{R}^{2n+1}$.*

(Comm. PDE, 2011)

Extension:

$$\tilde{\mathcal{L}} = \partial_t + (Av)\cdot\partial_x - \sum_{j,k=1}^{n} a_{jk}(t,x,v)\partial^2_{v_j v_k},$$

where $(t,x,v) \in U \subset \mathbb{R}^{2n+1}$, A is a non singular $n \times n$ constant matrix, $\big(a_{jk}(t,x,v)\big)$ is positive defined on U and belongs to $G^s(U)$.

Semi-linear Fokker-Planck equation:

$$\mathcal{L}u = \partial_t u + v\cdot\nabla_x u - a(t,x,v)\triangle_v u = F(t,x,v,u,\nabla_v u), \tag{1.6}$$

where $F(t,x,v,w,p)$ is nonlinear function of real variable (t,x,v,w,p), we have

Theorem 1.3. *Under the condition of Theorem 1.1, let u be a weak solution of the equation (1.6), such that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ and $\nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$, then*

$$u \in G^s(\mathbb{R}^{2n+1})$$

for any $s \geq 3$, if the nonlinear function $F(t,x,v,w,p) \in G^s(\mathbb{R}^{2n+2+n})$.

(JDE, 246 (2009),320-339)

Remark: *If the nonlinear term $F(t,x,v,w,p)$ is independent of p or F is in the form of $\nabla_v G(t,x,v,u)$, then it is enough to suppose in Theorem 1.2 the weak solution $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$.*

We consider now the semi-linear equation

$$\partial_t u + v\cdot\nabla_x u + a(-\triangle_v)^\sigma u = F(t,x,v;u), \tag{1.7}$$

where F is a nonlinear function of real variable (t,x,v,q). And we have the following Gevrey regularity for the weak solution.

Theorem 1.4. *Let $0 < \sigma < 1$ and $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\right\}$. Suppose $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$ is a weak solution of equation (1.7), then*

$$u \in G^s(\mathbb{R}^{2n+1})$$

for any $s \geq \frac{2}{\delta}$, provided the coefficient a is in $G^s(\mathbb{R}^{2n+1})$, $a(t,x,v) > 0$ and nonlinear function $F(t,x,v,q)$ lies in $G^s(\mathbb{R}^{2n+2})$.

(Comm. PDE, 2011)

2. Sub-elliptic estimates

Sharp Sub-elliptic Estimate:

Proposition 2.1. *$K \subset \mathbb{R}^{2n+1}$ compact. $\forall r \geq 0$, $\exists C_{K,r} > 0$, s.t. $\forall f \in C_0^\infty(K)$,*

$$\|f\|_r \leq C_{K,r}\{\ \|\mathcal{L}f\|_{r-2/3} + \|f\|_0\ \}. \tag{2.1}$$

Estimate Commutators of $\mathcal{L}$ with differential operators and cut-off functions:

Proposition 2.2. *$K \subset \mathbb{R}^{2n+1}$ compact, then $\forall r \geq 0$, $\exists C_{K,r} > 0$, $C_{K,r,\varphi} > 0$, s.t. $\forall f \in C_0^\infty(K)$,*

$$\|[\mathcal{L},\ D]f\|_r \quad \leq \quad C_{K,r}\{\ \|\mathcal{L}f\|_{r+1-2/3} + \|f\|_0\ \},$$

and

$$\|[\mathcal{L},\ \varphi]f\|_r \leq C_{K,r,\varphi}\{\ \|\mathcal{L}f\|_{r-1/3} + \|f\|_0\ \},$$

where $\varphi \in C_0^\infty(\mathbb{R}^{2n+1})$ and we denote by D the differential operators ∂_t, ∂_x or ∂_v.

Gevrey hypoellipticity of $\mathcal{L}$: Starting point due to M. Durand (1978):

Proposition 2.3. *Let P be a linear differential operator of second order with smooth coefficients in $\mathbb{R}^N_y$ and $\varrho, \varsigma > 0$ fixed. If $\forall r \geq 0$, $\forall K \subseteq \mathbb{R}^N$, $\forall \varphi \in C^\infty(\mathbb{R}^N)$, $\exists C_{K,r}$, $\exists C_{K,r}(\varphi)$, s.t. $\forall f \in C_0^\infty(K)$,*

$$\|f\|_r \leq C_{K,r}(\|Pf\|_{r-\varrho} + \|f\|_0), \tag{H_1}$$

$$\|[P,\ D_j]f\|_r \leq C_{K,r}(\|Pf\|_{r+1-\varsigma} + \|f\|_0), \tag{H_2}$$

$$\|[P,\ \varphi]f\|_r \leq C_{K,r}(\varphi)(\|Pf\|_{r-\varsigma} + \|f\|_0), \tag{H_3}$$

where

$$D_j = \frac{1}{i}\frac{\partial}{\partial y_j}, j = 1, 2, \cdots, N.$$

Then for $s \geq \max(1/\varsigma, 2/\varrho)$, P is $G^s(\mathbb{R}^N)$ hypoelliptic, provided the coefficients of P are in the class of $G^s(\mathbb{R}^N)$.

Proof of Theorem 1.1

Proposition 2.1 shows that the operator $\mathcal{L}$ satisfies the conditions (H_1) with $\varrho = 2/3$, Proposition 2.2 assures the conditions (H_2) and (H_3) with $\varsigma = 1/3$. Then $\max(1/\varsigma, 2/\varrho) = 3$, $\mathcal{L}$ is $G^s(\mathbb{R}^{2n+1})$ hypoelliptic for $s \geq 3$, and we have proved Thereom 1.1.

Proof of Theorem 1.2

The proof of Theorem 1.2 depends on following sub-elliptic estimate:

There exists a compact $K \subset \mathbb{R}^{2n+1}$, and for $\forall r \geq 0$, $\exists C_{K,r} > 0$, s.t. $\forall f \in C_0^\infty(K)$,

$$\|f\|_r \leq C_{K,r}\{\ \|\mathcal{P}f\|_{r-\delta} + \|f\|_0\ \}, \tag{2.2}$$

where $\delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\right\}$.

Remark: *Very recently, N. Lerner, Y. Morimoto and K. Pravda-Starov proved following optimal sub-elliptic estimate for the operator $\mathcal{P}$:*

$$\|f\|_r \leq C_{K,r}\{\ \|\mathcal{P}f\|_{r-\delta_1} + \|f\|_0\ \}, \tag{2.3}$$

where $\delta_1 = \frac{2\sigma}{2\sigma+1}(> \delta = \max\left\{\frac{\sigma}{4},\ \frac{\sigma}{2} - \frac{1}{6}\right\})$, which implies the optimal hypoellipticity for the operator $\mathcal{P}$ would be G^s-regular in $\mathbb{R}^{2n+1}$ for all $s \geq \frac{2}{\delta_1}$. This coincides with the result of Theorem 1.1 in case of $\sigma = 1$.

3. Gevrey regularity of nonlinear equations

Let $u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$, a weak solution of (1.6), we need to prove $u \in C^\infty(\mathbb{R}^{2n+1})$, and then to prove $u \in G^s(\mathbb{R}^{2n+1})$

Proposition 3.1. *Let u be a weak solution of (1.6) such that $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1})$. Then u is in $C^\infty(\mathbb{R}^{2n+1})$.*

Proof of Proposition 3.1 depends on following results:

Lemma 3.2. *Let $F(t,x,v,w,p) \in C^\infty(\mathbb{R}^{2n+2+n})$ and $r \geq 0$. If $u, \nabla_v u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$, then $F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$, and*

$$\big\|\phi_1 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\big\|_r \leq \bar{C}\left\{ \|\phi_2 u\|_r + \|\phi_2 \nabla_v u\|_r \right\}, \tag{3.1}$$

where $\phi_1,\ \phi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\phi_2 = 1$ on the support of ϕ_1, and $\bar{C}$ is a constant depending only on $r,\ \phi_1,\ \phi_2$.

Remark. *If the nonlinear term F is independent of p or in the form of*

$$\nabla_v(F(t,x,v,u)),$$

Then that $u \in L^\infty_{loc}(\mathbb{R}^{2n+1}) \cap H^r_{loc}(\mathbb{R}^{2n+1})$ yields $F\big(\cdot, u(\cdot),\ \nabla_v u(\cdot)\big) \in H^r_{loc}(\mathbb{R}^{2n+1})$.

Lemma 3.3. *Let $u,\ \nabla_v u \in H^r_{loc}(\mathbb{R}^{2n+1}),\ r \geq 0$. Then we have*

$$\|\varphi_1 \nabla_v u\|_r \leq C\, \|\varphi_2 u\|_r\,, \tag{3.2}$$

where $\varphi_1,\ \varphi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\varphi_2 = 1$ on the support of φ_1, and C is a constant depending only on $r,\ \varphi_1,\ \varphi_2$.

Proof of Proposition 3.1:

In fact, from the sub-elliptic estimate (2.1) and the fact $\mathcal{L}u(\cdot) = F(\cdot, u(\cdot), \nabla_v u(\cdot))$, it then follows

$$\|\psi_1 u\|_{r+2/3} \leq \bar{C}\{ \|\psi_2 F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big)\|_r + \|\psi_2 u\|_0 \}, \tag{3.3}$$

where $\psi_1,\ \psi_2 \in C_0^\infty(\mathbb{R}^{2n+1})$ and $\psi_2 = 1$ on the support of ψ_1. Combining (3.1), (3.2) and (3.3), we have $u \in H^\infty_{loc}(\mathbb{R}^{2n+1})$ by standard iteration. This completes the proof of Proposition 3.1.

Now starting from the smooth solution, we can then prove the solution has Gevrey Regularity. It suffices to show the Gevrey regularity in an open unit ball

$$\Omega = \{(t,x,v) \in \mathbb{R}^{2n+1} : t^2 + |x|^2 + |v|^2 < 1\}.$$

Set

$$\Omega_\rho = \big\{(t,x,v) \in \Omega : \big(t^2 + |x|^2 + |v|^2\big)^{1/2} < 1 - \rho\big\}, \qquad 0 < \rho < 1.$$

Let U be an open subset of $\mathbb{R}^{2n+1}$. Denote by $H^r(U)$ the space consisting of the functions which are defined in U and can be extended to $H^r(\mathbb{R}^{2n+1})$. Define

$$\|u\|_{H^r(U)} = \inf\big\{\|\tilde{u}\|_{H^s(\mathbb{R}^{n+1})} : \tilde{u} \in H^s(\mathbb{R}^{2n+1}), \tilde{u}|_U = u\big\}.$$

We denote $\|u\|_{r,U} = \|u\|_{H^r(U)}$, and

$$\|\partial_x^j u\|_r = \sum_{|\beta|=j} \|D_x^\beta u\|_r, \quad x \in \mathbb{R}^m.$$

In order to treat the nonlinear term F on the right hand side of (1.6), we need following two lemmas.

Lemma 3.4. *Let $r > (2n+1)/2$ and $u_1, u_2 \in H^r(\mathbb{R}^{2n+1})$, Then $u_1 u_2 \in H^r(\mathbb{R}^{2n+1})$, moreover*

$$\|u_1 u_2\|_r \leq \tilde{C}\|u_1\|_r \|u_2\|_r, \tag{3.4}$$

where $\tilde{C}$ is a constant depending only on $n,\ r$.

Lemma 3.5. *Let M_j be a sequence of positive numbers and for some $B_0 > 0$, M_j satisfy following monotonicity conditions*

$$\frac{j!}{i!(j-i)!} M_i M_{j-i} \leq B_0 M_j, \ (i = 1, 2, \cdots, j; \ j = 1, 2, \cdots). \tag{3.5}$$

Suppose $F(t, x, v, u, p)$ satisfies (for $j, \ m + l \geq 2$)

$$\| \left(\partial^j_{t,x,v} D^l_u \partial^m_p F \right) (\cdot, u(\cdot), \nabla_v u(\cdot)) \|_{r+n+1,\Omega} \leq C_1^{j+l+m} M_{j-2} M_{m+l-2}, \tag{3.6}$$

where r is a real number, satisfying $r + n + 1 > (2n + 1)/2$. Then there exist two constants C_2, C_3 such that for any H_0, H_1 satisfying H_0, $H_1 \geq 1$ and $H_1 \geq C_2 H_0$, if $u(t, x, v)$ satisfies following conditions

$$\|\partial^j_{t,x,v} u\|_{r+n+1,\Omega_{\tilde\rho}} \leq H_0, \quad 0 \leq j \leq 1, \tag{3.7}$$

$$\|\partial^j_{t,x,v} u\|_{r+n+1,\Omega_{\tilde\rho}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq N, \tag{3.8}$$

$$\|D_v \partial^j_{t,x,v} u\|_{r+n+1,\Omega_{\tilde\rho}} \leq H_0 H_1^{j-2} M_{j-2}, \quad 2 \leq j \leq N, \tag{3.9}$$

then for all α with $|\alpha| = N$,

$$\left\| \psi_N D^\alpha \big[F\big(\cdot, u(\cdot), \nabla_v u(\cdot)\big) \big] \right\|_{r+n+1} \leq C_3 H_0 H_1^{N-2} M_{N-2}, \tag{3.10}$$

where $\psi_N \in C_0^\infty(\Omega_{\tilde\rho})$ is an arbitrary function.

Key Estimate:

Proposition 3.6. *Let $s \geq 3$. Suppose $u \in C^\infty(\bar\Omega)$ is a solution of (1.6), and $a(t, x, v) \in G^s(\mathbb{R}^{2n+1})$, $F(t, x, v, w, p) \in G^s(\mathbb{R}^{2n+2+n})$ and $a(t, x, v) \geq c_0 > 0$. Then there exits a constant A such that for any $r \in [0, 1]$ and any $N \in \mathbb{N}$, $N \geq 3$,*

$$(E)_{r,N} \qquad \|D^\alpha u\|_{r+n+1,\Omega_\rho} + \|D_v D^\alpha u\|_{r-1/3+n+1,\Omega_\rho}$$
$$\leq \frac{A^{|\alpha|-1}}{\rho^{s(|\alpha|-3)}} \big((|\alpha| - 3)!\big)^s (N/\rho)^{sr}, \quad \forall \ |\alpha| = N, \ \forall \ 0 < \rho < 1.$$

From Proposition 3.6, we have immediately

Corollary 3.7. *Under the same assumption as Proposition 3.6, we have $u \in G^s(\Omega)$.*

In fact, for any compact subset K of Ω, we have $K \subset \Omega_{\rho_0}$ for some ρ_0, $0 < \rho_0 < 1$. For any α, $|\alpha| \geq 3$, letting $r = 0$ in $(E)_{r,N}$, we have

$$\|D^\alpha u\|_{L^2(K)} \leq \|D^\alpha u\|_{n+1,\Omega_{\rho_0}} \leq \frac{A^{|\alpha|-1}}{{\rho_0}^{s(|\alpha|-3)}} \big((|\alpha| - 3)!\big)^s \leq \big(\tfrac{A}{{\rho_0}^s}\big)^{|\alpha|+1} (|\alpha|!)^s.$$

This completes the proof of Corollary 3.7

4. Smoothness effects of solutions for Cauchy problem of the spatially homogeneous Landau equation

Let

$$\begin{cases} \partial_t f = \nabla_v \cdot \big\{ \int_{\mathbb{R}^3} a(v - v_*)[f(v_*) \nabla_v f(v) - f(v) \nabla_v f(v_*)] dv_* \big\}, \\ f(0, v) = f_0(v), \end{cases} \tag{4.1}$$

where $f(t, v) \geq 0$ stands for the density of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$, and (a_{ij}) is a nonnegative symmetric matrix given by

$$a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right) |v|^{\gamma+2}. \tag{4.2}$$

We only consider here the condition $\gamma \in [0, 1]$. It's called the hard potential case when $\gamma \in]0, 1]$ and the Maxwellian molecules case when $\gamma = 0$.

Set $c = \sum_{i,j=1}^{3} \partial_{v_i v_j} a_{ij} = -2(\gamma+3)\,|v|^{\gamma}$ and

$$\bar{a}_{ij}(t,v) = (a_{ij} * f)(t,v) = \int_{\mathbb{R}^3} a_{ij}(v-v_*) f(t,v_*) dv_*, \quad \bar{c} = c * f.$$

Then the Cauchy problem (4.1) can be rewritten as the following form,

$$\left\{ \begin{array}{l} \partial_t f = \sum_{i,j=1}^{3} \bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f, \\ f(0,v) = f_0(v). \end{array} \right. \tag{4.3}$$

This is a non-linear diffusion equation, and the coefficients $\bar{a}_{ij}, \bar{c}$ depend on the solution f.

Denote by $M(f(t))$, $E(f(t))$ and $H(f(t))$ respectively the mass, energy and entropy of the function $f(t)$, i.e.,

$$M(f(t)) = \int_{\mathbb{R}^3} f(t,v)\, dv, \quad E(f(t)) = \frac{1}{2} \int_{\mathbb{R}^3} f(t,v)\, |v|^2 \, dv,$$

$$H(f(t)) = \int_{\mathbb{R}^3} f(t,v) \log f(t,v)\, dv,$$

and

$$\|\partial^{\alpha} f(t,\cdot)\|_{L^p_s}^p = \|\partial^{\alpha} f(t)\|_{L^p_s}^p = \int_{\mathbb{R}^3} |\partial^{\alpha} f(t,v)|^p \left(1+|v|^2\right)^{s/2} dv, \text{ for } p \geq 1,$$

$$\|f(t,\cdot)\|_{H^m_s}^2 = \|f(t)\|_{H^m_s}^2 = \sum_{|\alpha| \leq m} \|\partial^{\alpha} f(t,\cdot)\|_{L^2_s}^2 .$$

We have

Theorem 4.1. *Let f_0 be the initial datum with finite mass, energy and entropy and f be any solution of the Cauchy problem (4.3) such that for all t_0, t_1 with $0 < t_0 < t_1 < +\infty$, and all integer $m \geq 0$,*

$$\sup_{t \in [t_0,t_1]} \|f(t,\cdot)\|_{H^m_{\gamma}} < +\infty. \tag{4.4}$$

Then for any number $\sigma > 1$, we have $f(t,\cdot) \in G^{\sigma}(\mathbb{R}^3)$ for all time $t > 0$.

(Acta Math. Scientia, 29B(3), 673-686(2009), Special volume for Prof. WU Wenjun 90th birthday)

Remark. *From Desvillettes-Villani [Comm PDE, Vol.25 (2000)], if $f_0 \in L^1_{2+\delta}$ with $\delta > 0$), then the condition (4.4) would be satisfied (i.e. C^{∞} smoothness effect).*

Furthermore, we have

Theorem 4.2. *Let f_0 be the initial datum with finite mass, energy and entropy and f be any solution of the Cauchy problem (4.3) such that for all t_0, t_1 with $0 < t_0 < t_1 < +\infty$, and all integer $m \geq 0$,*

$$\sup_{t \in [t_0,t_1]} \|f(t,\cdot)\|_{H^m_{\gamma}} < +\infty. \tag{4.5}$$

Then for all time $t > 0$, $f(t,\cdot)$, as a real function of v variable, is analytic in $\mathbb{R}^3_v$. Moreover, for all time $t_0 > 0$, there exists a constant $c_0 > 0$, depending only on M_0, E_0, H_0, γ and t_0, such that for all $t \geq t_0$,

$$\left\| e^{c_0(-\triangle_v)^{\frac{1}{2}}} f(t,\cdot) \right\|_{L^2(\mathbb{R}^3)} \leq C(t+1),$$

where C is a constant depending only on M_0, E_0, H_0, γ and t_0.

(JDE, 248, 77-94 (2010))

Remark. *In Maxwellian molecules case, Morimoto-Xu (JDE, 247(2009), 596–617) even proved the ultra-analyticity for the solution of the Cauchy problem (4.3).*

Ultra-Analyticity:

Definition. $f \in A^s(\Omega)$ for $0 < s < +\infty$, if $f \in C^\infty(\Omega)$ and $\exists C > 0$, $N_0 > 0$, such that $\forall \alpha \in \mathbb{N}^n$, $|\alpha| \geq N_0$,

$$\|\partial^\alpha f(x)\|_{L^2(\Omega)} \leq C^{|\alpha|+1}(|\alpha|!)^s.$$

If $\Omega = \mathbb{R}^n$, then $f \in A^s(\mathbb{R}^n)$ iff

$$e^{c_0(-\triangle)^{1/2s}}(\partial^{\beta_0} f) \in L^2(\mathbb{R}^n), \quad \exists c_0 > 0,\ \beta_0 \in \mathbb{N}^n.$$

Thus

1) $A^{s_1} \subset A^{s_2}$ if $s_1 < s_2$;
2) $A^s = G^s$, for $s > 1$;
3) $A^1 =$ Analytic class;
4) A^s is ultra-analytic for $0 < s < 1$;
5) Any polynomial $P(x)$ is ultra-analytic for any $s > 0$.

Examples.

1) Heat operator:

$$\begin{cases} \partial_t u - \triangle_x u = 0, \ \ x \in \mathbb{R}^n,\ t > 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^n). \end{cases}$$

Then $u(t, \cdot) = e^{t\triangle_x} u_0 \in A^{\frac{1}{2}}(\mathbb{R}^n)$ for $t > 0$.

2) Kolmogorov operator:

$$\begin{cases} \partial_t u + v \cdot \partial_x u - \triangle_v u = 0, \ \ (x, v) \in \mathbb{R}^{2n},\ t > 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^{2n}). \end{cases}$$

Then $\hat{u}(t, \eta, \xi) = e^{-\int_0^t |\xi + s\eta|^2 ds} \hat{u_0}(\eta, \xi + t\eta)$, i.e. for some $c_0 > 0$, we have $u(t, \cdot, \cdot) = e^{c_0(t\triangle_v + t^3\triangle_x)} u_0 \in A^{\frac{1}{2}}(\mathbb{R}^{2n})$ for $t > 0$.

Remark: *It is surprise to say the ultra-analytic smoothness effect phenomenon for the Kolmogorov equation is similar to* $2n$*-dimensional heat equation. That means, in some sense, the operator* $v \cdot \partial_x - \triangle_v$ *is equivalent to* $2n$*-dimensional Laplace operator* $\triangle_{x,v}$ *by time evolutions. But the operator* $\partial_t + v \cdot \partial_x$ *is only a transport operator with respect to spatial variables.*

3) Generalized Kolmogorov operators:

$$\begin{cases} \partial_t u + v \cdot \partial_x u + (-\triangle_v)^\sigma u = 0, \ \ (x, v) \in \mathbb{R}^{2n},\ t > 0, \\ u|_{t=0} = u_0 \in L^2(\mathbb{R}^{2n}), \quad 0 < \sigma < +\infty. \end{cases}$$

Then $\hat{u}(t, \eta, \xi) = e^{-\int_0^t |\xi + s\eta|^{2\sigma} ds} \hat{u_0}(\eta, \xi + t\eta)$, i.e. for some $c_0 > 0$, we have $u(t, \cdot, \cdot) = e^{-c_0(t(-\triangle_v)^\sigma + t^{2\sigma+1}(-\triangle_x)^\sigma)} u_0 \in A^{\frac{1}{2\sigma}}(\mathbb{R}^{2n})$ for $t > 0$.

Thus, for $\sigma > \frac{1}{2}$, we can get ultra-analytic smoothing effect in $A^{\frac{1}{2\sigma}}(\mathbb{R}^{2n})$.

Comparing Results with Heat Operator:

1) Heat operator $H = \partial_t - \triangle_{x,v}$ is $G^2(\mathbb{R}^{2n+1})$ hypoelliptic;

2) Fokker-Planck operator $\mathcal{L} = \partial_t + v \cdot \partial_x f - a(t, x, v)\triangle_v$ is $G^3(\mathbb{R}^{2n+1})$ hypoelliptic, cf. Chen-Li-Xu's result in 2008 (JDE, 246 (2009), 320-339);

3) The operator $\mathcal{P} = \partial_t + v \cdot \partial_x + a(t, x, v)(-\triangle_v)^\sigma,\ 0 < \sigma < 1$, is $G^{\frac{2}{\delta}}(\mathbb{R}^{2n+1})$ hypoelliptic, where $\delta = \max\{\frac{\sigma}{4}, \frac{\sigma}{2} - \frac{1}{6}\}$, cf. Chen-Li-Xu's result in 2009 (Comm. in PDE, 2011); and is $G^{\frac{2}{\delta_1}}(\mathbb{R}^{2n+1})$ hypoelliptic, where $\delta_1 = \frac{2\sigma}{2\sigma+1}$, which can be deduced by Lerner-Morimoto-Pravda-Starov's recent result on optimal sub-elliptic estimate for the operator $\mathcal{P}$.

Optimal Smoothing Effect for Cauchy Problems:

1) For Heat Equation, we have $A^{\frac{1}{2}}$ ultra-analytic effect for $t > 0$, i.e. $u(t, \cdot) \in A^{\frac{1}{2}}(\mathbb{R}^n)$ for $t > 0$.

2) For Fokker-Planck Equation (with $a(t, x, v) = 1$, i.e. Kolmogorov Equation), we have also $A^{\frac{1}{2}}$ ultra-analytic effect for $t > 0$, i.e. $u(t, \cdot, \cdot) \in A^{\frac{1}{2}}(\mathbb{R}^{2n})$ for $t > 0$.

3) For spatially homogeneous Landau equation:

$$\begin{cases} \partial_t f = \sum_{i,j=1}^{3} \bar{a}_{ij} \partial_{v_i v_j} f - \bar{c} f, \\ f(0, v) = f_0(v). \end{cases}$$

In Maxwellian molecules case, we have $A^{\frac{1}{2}}$ ultra-analytic effect for $t > 0$, i.e. in the case of $a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right) |v|^2$ (2nd order polynomial in v), the solution of the Cauchy problem $u(t, \cdot) \in A^{\frac{1}{2}}(\mathbb{R}^n)$ for $t > 0$. In hard potential case, we have A^1-analytic effect for $t > 0$, i.e. in the case of $a_{ij}(v) = \left(\delta_{ij} - \frac{v_i v_j}{|v|^2}\right) |v|^{\gamma+2}$, for $\gamma \in]0, 1]$, the solution of the Cauchy problem $u(t, \cdot) \in A^1(\mathbb{R}^n)$ for $t > 0$.

5. Regularity of Solutions for Subelliptic Monge-Ampère Equations

Let us consider following real Monge-Ampère equation:

$$F(u_{ij}) = \det(u_{ij}) = k(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{5.1}$$

where $u_{ij} = \frac{\partial^2 u}{x_i x_j}(x)$. We assume that

1) $H = (u_{ij})$ is a non-negative defined matrix on Ω;

2) Hypersurface $S = \{(x, u(x)); x \in \Omega\}$ is convex (but not strict);

3) $k(x) \geq 0$ on Ω.

Well-known results:

1) $k > 0$, non-degenerate case, regularity is well-known (Equation (5.1) is elliptic, c.f. Caffarelli-Nirenberg-Spruck, CPAM, 37(1984) for Dirichlet problem). Then $u \in C^\infty$ if $k \in C^\infty$ and u is analytic if k is analytic.

2) In degenerate case,

$$\Sigma_k = \{x \in \Omega; k(x) = 0, \nabla k(x) = 0\} \neq \emptyset, \tag{5.2}$$

the regularity problem would be complicated. (Existence and uniqueness have been studied by Guan-Trudinger-Wang, Acta Math, 182 (1999).)

3) Monge-Ampère equation has a C^∞ convex local solution if the order of degenerate point for the smooth coefficient k is finite (c.f. Hong-Zuily, Invent. Math. 89, 645-661 (1987)).

4) For regularity result, Zuily proved that, for the degenerate Monge-Ampère equation, if the solution $u \in C^\rho$ for $\rho > 4$, then u will be C^∞ smooth (c.f. Zuily, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, 11(4), 529-554 (1988); we would say more for this result in details).

5) In case of $k \geq 0$, a C^∞-smooth function, but vanishes in Ω, in general, the optimal regularity of solutions is $C^{1,1}$ (c.f. Guan, Duke Math J., 86 (1997)).

Main difficulties: 1) Fully nonlinear; 2) Degeneracy.

Basic Question: When is a solution u of (5.1) smooth, or better than $C^{1,1}$?

One may expect u to be smooth if the decay of k near its null set is under control, say of finite type. Unfortunately, this is not true:

Example 1. Function $u(x) = |x|^{2+\frac{2}{n}} \in C^{2,2/n}$ ($\notin C^3$) solves the equation (5.1) with polynomial data $k(x) = c_n |x|^2$ (analytic, vanishing at only one point of order 2).

What is wrong with the Example 1 is that the mean curvature of hypersurface $(x, u(x))$ is vanishing at the point $k = 0$. This suggests that we should only expect higher regularity of the solution u of (5.1) away from the planar points of the hypersurface $(x, u(x))$.

Example 2. Zuily (1988) extended above counter-example to the general cases, i.e. for any integer $l \geq 2$, $\exists k(x) \geq 0$, analytic, such that the Monge-Ampère equation (5.1) admit a solution in C^l, but not in C^{l+1} (There are a lot of examples for non-smooth solutions of Dirichlet problems).

Example 3. If $n = 2$, $k(x, y) = x^2$, then

$$u(x, y) = \frac{1}{12}x^4 + \frac{1}{2}y^2, \quad \text{(i.e. } u_{yy} > 0)$$

is a analytic solution .

So it is reasonable to find the conditions on Σ_k and also on the geometry of $S = \{(x, u(x)); x \in \Omega\}$ to guarantee the regularity of the weak solutions.

The result of Zuily (1988):
We consider the linearized operator,

$$P_u = \sum_{i,j=1}^{n} \frac{\partial F}{\partial u_{ij}}(u_{lk})\partial^2_{ij}, \tag{5.3}$$

where $\frac{\partial F}{\partial u_{ij}}(u_{lk})$ is the co-factor of u_{ij} in the matrix H. Then the co-matrix $\tilde{H}$ of H is also non-negative defined, so the second order operator P_u is degenerate elliptic if $\Sigma_k \neq \emptyset$. Denote $\tilde{l}_\alpha(x) = l_\alpha(u_{lk}(x))$ the α-line of the matrix $\tilde{H}$, then Zuily (1988) proved that

Proposition 5.1. *Let $k(x) \in C^\infty$, $k \geq 0$, and $u(x) \in C^{4,\delta}$ is a solution of Monge-Ampère equation (5.1). Suppose that for any $x \in \Sigma_k$, $\exists 1 \leq \alpha \leq n$, such that*

$$< \nabla^2 k(x)\tilde{l}_\alpha, \tilde{l}_\alpha > \neq 0, \tag{5.4}$$

then the solution $u(x) \in C^\infty$.

Remarks:

(i) No conditions required in out of Σ_k, since the equation is elliptic on $\Omega \backslash \Sigma_k$.

(ii) If $\nabla^2 k(x)$ is positive defined, then the condition (5.4) is satisfied. So any $C^{4,\delta}$ solution would be C^∞ smooth. But Example 1 shows that the initial regularity of the solution is important.

(iii) In 2-dimensional case, in Example 3, $k(x, y) = x^2$, the condition (5.4) means $u_{yy}(0, y) \neq 0$ (in fact, $u_{yy}(0, y) = 1 > 0$).

6. Main Result for Monge-Ampère Equations

Let us consider regularity of solution to following two-dimensional Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = k(x, y), \quad (x, y) \in \Omega \subset \mathbb{R}^2, \tag{6.1}$$

where Ω is neighborhood of origin, k is a nonnegative function.

By using the classic partial Legendre transformation, the equation (6.1) can be deduced as following divergence form quasi-linear equation:

$$\partial_{ss}w(s, t) + \partial_t \{k(s, w(s, t))\partial_t w(s, t)\} = 0. \tag{6.2}$$

In the 70's, there were many works on studying the hypoellipticity of linear degenerate elliptic equations (i.e. Hörmander's operators). Baouendi-Goulaouic (1971) gave following example in $\mathbb{R}^3$:

$$Pu = (\partial_x^2 + \partial_y^2 + y^2\partial_z^2)u = f. \tag{6.3}$$

We know:

- The operator P is C^∞ hypoelliptic.

- The operator P is not analytical hypoelliptic (Baouendi-Goulaouic even proved that there is a analytic function f, such that the solution $u \in G^s$ with $s > 2$, but not belongs to G^s for $s < 2$).
- Derridj-Zuily (1972) proved the Gevrey regularity for such kind of Hörmander's operators (also in 2-dimensional case, the operator $\partial_x^2 + x^2\partial_y^2$ is G^s hypoelliptic for any $s \geq 1$).

For Monge-Ampère equation, in order to get higher regularity, we need following reasonable conditions:
(I) Condition for k (i.e. condition on Σ_k):
$k \geq 0$, $C^\infty(\mathbb{R}^2)$ smooth,
k vanishes in Ω with finite order, i.e., $\exists c > 1$, such that

$$c^{-1}(x^{2l} + Ay^{2m}) \leq k(x, y) \leq c(x^{2l} + Ay^{2m}),$$

where $A \geq 0$, and $l \leq m$ are two nonnegative integers.
(II) Non-planar Condition:

$$u_{yy} \geq c_0 > 0 \qquad \text{in } \Omega.$$

This means the hypersurface $S = \{(x, u(x)); x \in \Omega\}$ does not have planar points, or say that one principal curvature of the solution u is strictly positive (this is a geometry condition).

Well-known result:

Proposition 6.1. *(P. Guan, Adv. Math., 132(1997)) Let u be a $C^{1,1}$ weak solution of (5.1), then under the conditions (I) and (II), we have $u \in C^\infty(\Omega)$.*

Remark: *There have been some works on degenerate nonlinear elliptic equations in connection with Bony's theory of paradifferential operators. Under some initial smoothness assumptions on the solution with some subelliptic estimates, Xu (Comm. PDE, 11 (1986)) proved the solution is C^∞ by paradifferential calculus.*

Question: Is it the best possible for the regularity of solution for degenerate Monge-Ampère equation to be C^∞ smooth?

Theorem 6.2. *(***Main result***) (Chen-Li-Xu, 2009). Let u be a $C^{1,1}$ weak solution of the Monge-Ampère equation (5.1), then under the conditions (I) and (II), we have $u \in G^{l+1}(\Omega)$, provided $k \in G^{l+1}(\mathbb{R}^2)$.*

Remarks:

- *If $k(x, y) \in G^\sigma(\mathbb{R}^2)$ with $\sigma \geq l + 1$, then we have $u \in G^\sigma(\Omega)$.*
- *The result can be extended to higher dimensional cases, and the case of generalized Monge-Ampère equation*

$$\det D^2 u = k(x, u, Du), \qquad x \in \Omega.$$

- *The Gevrey index $l + 1$ seems the best possible, since, in case of $l = 0$, the equation (6.1) is elliptic and $u \in G^1(\Omega) = \mathcal{A}(\Omega)$, i.e., the solution has analytic regularity, which coincides with the well-known regularity result for nonlinear elliptic equation.*
- *If $k(x, y)$ is independent of second variable y (i.e. $A = 0$), the equation (6.2) is linear, then Derridj-Zuily (1972) proved that the optimal regularity result is that the solution $u \in G^s$, $s \geq 1$, provided $k \in G^s$.*

7. Idea of Proof.

By using the partial Legendre transformation, the equation (6.1) can be translated to the following divergence form quasi-linear equation

$$\partial_{ss} w(s, t) + \partial_t \{k(s, w(s, t))\partial_t w(s, t)\} = 0. \tag{7.1}$$

Thus we need to prove

1) Gevery regularity for the equation (7.1);
2) Gevrey regularity will be kept to be invariant under the partial Legendre transformation.

Partial Legendre transformation (or called semi-spherical mapping): $T:(x,y)\longrightarrow(s,t)$ by setting

$$\left\{\begin{array}{lcl} s & = & x, \\ t & = & u_y. \end{array}\right. \tag{7.2}$$

It is easy to verify that

$$J_T=\begin{pmatrix} s_x & s_y \\ t_x & t_y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ u_{xy} & u_{yy} \end{pmatrix},$$

and

$$J_T^{-1}=\begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ -\frac{u_{xy}}{u_{yy}} & \frac{1}{u_{yy}} \end{pmatrix}.$$

Thus if $u\in C^{1,1}$ and $u_{yy}\geq c_0>0$ in Ω, then the transformations

$$T:\Omega\longrightarrow T(\Omega),\quad T^{-1}:T(\Omega)\longrightarrow\Omega$$

are $C^{0,1}$-differmorphisms, and if $u\in C^\infty$ then T is C^∞-differmorphisms.

Main result will be deduced by following propositions:

Proposition 7.1. *If $u(x,y)$ is a smooth solution of the Monge-Ampère equation (6.1) and $u_{yy}\geq c_0>0$ in Ω, then $y(s,t)\in C^\infty(T(\Omega))$ satisfying the equation*

$$\partial_{ss}y+\partial_t\Big\{k(s,y(s,t))\partial_t y\Big\}=0. \tag{7.3}$$

Proposition 7.2. *Suppose that $w(s,t)\in C^\infty(\bar{B})$ is a solution to the quasi-linear equation (7.1), and that $k\in G^{\ell+1}(\mathbb{R}^2)$. Then $w\in G^{\ell+1}(B)$ (where $B=\{(x,t)|s^2+t^2<1\}$).*

Proposition 7.3. *If $k\in G^{\ell+1}(\Omega)$ and $y(s,t)\in G^{\ell+1}(T(\Omega))$, then $u(x,y)\in G^{\ell+1}(\Omega)$.*

The result of Proposition 7.3 implies that the Gevrey regularity will be invariant under the partial Legendre transformation.

Thus we have

- From Guan's result $\Rightarrow u\in C^\infty(\Omega)$.
- From Proposition 7.1 and Proposition 7.2 $\Rightarrow w(s,t)\in G^{l+1}(T(\Omega))$.
- From Proposition 7.3$\Rightarrow u\in G^{l+1}(\Omega)$ (Main Result).

Proof of Proposition 7.1 can be deduced directly by Lemma 19 of Guan's paper (Adv. Math. 1997). So it remains to prove Proposition 7.2 and Proposition 7.3 (we omitted here).