SEMICLASSICAL WIDTH OF RESONANCES CREATED BY HOMOCLINIC ORBITS

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Abstract. Semiclassical width of resonances of the Schrödinger operator in a neighborhood of a fixed energy is closely related with the set of trapped trajectories of the underlying classical mechanics.

We consider the case where the trapped set consists of homoclinic trajectories associated with a hyperbolic fixed point, and we obtain that the width of resonances is greater than a constant multiple of the semiclassical parameter if the fixed point is anisotropic or the dimension of the trapped set is smaller than the space dimension.

The key of the proof is the propagation formula of microlocal solutions near a hyperbolic fixed point from the incoming stable manifold to the outgoing one, established in [4].

This note is a brief summary of the forthcoming paper [5].

1. Introduction

We consider the semiclassical Schrödinger operator in $\mathbb{R}^n$

$$P := -h^2\Delta + V(x) = -h^2\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2} + V(x), \tag{1.1}$$

where h is a small positive parameter and $V(x)$ is a potential satisfying:

(A0) $V(x)$ is real-valued on $\mathbb{R}^n$ and analytic in a sector

$$\mathcal{S} = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| \le \tan\theta_0 \langle \operatorname{Re} x\rangle\},$$

for some positive θ_0. Moreover $V(x)$ tends to 0 as x tends to ∞ in $\mathcal{S}$.

This condition enables us to define resonances of the operator P in the complex sector $C_\theta := \{E \in \mathbb{C}\setminus\{0\};\ \arg E \in (-2\theta, 0)\}$, $0 < \theta < \theta_0$, of the spectral parameter E, as eigenvalues of the non self-adjoint operator $P_\theta = U_\theta P U_{-\theta}$, where $[U_\theta f](x) := e^{i\frac{n\theta}{2}} f(e^{i\theta}x)$ for all $x \in \mathbb{R}^n$. The set of resonances $\Gamma_\theta(h)$ in C_θ is independent of the angle θ in the sense that $\Gamma_\theta(h) = \Gamma_{\theta'}(h)$ in C_θ for $\theta < \theta' < \theta_0$.

On the other hand, we consider the corresponding classical mechanics described by the classical Hamiltonian $p(x,\xi) := \xi^2 + V(x)$ for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Let

$$H_p := \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi = 2\xi \cdot \partial_x - \nabla V(x) \cdot \partial_\xi,$$

be the Hamiltonian vector field associated to p. Integral curves $t \mapsto \exp(tH_p)(x,\xi)$ of H_p are called classical trajectories or bicharacteristic curves, and p is constant along such curves. For a fixed positive energy E_0, we define the *trapped set*:

$$K(E_0) = \{(x,\xi) \in p^{-1}(E_0);\ t \mapsto \exp(tH_p)(x,\xi) \text{ is bounded}\}.$$

The geometry of the classical mechanics near the set $K(E_0)$ is closely related to the asymptotic distribution of resonances in a complex neighborhood of E_0. In particular, the "smallness" of $K(E_0)$, or in other words the "weakness" of the trap, is reflected to a large imaginary part of resonances, which implies the short life time of the quantum particles. Among many important results about this subject, we refer only to a few of those concerning resonances with large imaginary part:

First, if $K(E_0)$ is empty, there exists a constant $\varepsilon > 0$ such that there is no resonance in the h-independent domain $(E_0 - \varepsilon, E_0 + \varepsilon) - i(0,\varepsilon)$ for sufficiently small h (see [2] and [12]). If the potential is only C^∞ except near the infinity, there is no resonance in $(E_0 - \varepsilon, E_0 + \varepsilon) - i(0, Ch \log \frac{1}{h})$ for any $C > 0$ and sufficiently small h (see [14]).

In case where $K(E_0)$ consists of a hyperbolic fixed point (x_0, ξ_0), which occurs when the potential V presents a unique global non-degenerate maximum at a point $x = x_0$, the precise asymptotic distribution of resonances was obtained independently by Briet, Combes and Duclos [3], and Sjöstrand [16]. In fact, they proved that there exists a constant $\delta > 0$ such that for any $C > 0$, there is no resonance in $(E_0 - Ch, E_0 + Ch) - i(0, \delta h)$ for sufficiently small h. Here δ is any constant smaller than $\frac{1}{2}\sum_{j=1}^n \lambda_j$, where $\lambda_1, \dots, \lambda_n$ are the positive eigenvalues of $\sqrt{-2V''(x_0)}$, or equivalently, $\pm\lambda_1, \dots, \pm\lambda_n$ are the eigenvalues of the linearization of H_p at the fixed point.

In case where $K(E_0)$ consists of a hyperbolic periodic trajectory, which may occur when V has two convex bumps, the precise asymptotic distribution of resonances was obtained by Gérard and Sjöstrand (see [10]), and in particular they proved that there exist constants $\delta, \varepsilon > 0$ such that there is no resonance in $(E_0 - \varepsilon, E_0 + \varepsilon) - i(0, \delta h)$ for sufficiently small h. Here the optimal δ is determined by the eigenvalues of the linearized Poincaré map associated to the periodic trajectory.

If the number of bumps is more than 2, the trapped set $K(E_0)$ becomes a fractal set. Under the condition that the flot is hyperbolic on $K(E_0)$ and that this set is "sufficiently small" (more precisely, they make an assumption on the *topological pressure* of $K(E_0)$), Nonnenmacher and Zworski (see [15]) proved that the width of resonances is also greater than δh, with precise δ.

We will assume that $K(E_0)$ consists of homoclinic trajectories, i.e. a hyperbolic fixed point (x_0, ξ_0) and trajectories that tend to this point as time tends to both $+\infty$ and $-\infty$.

This situation is realized when, for example, the potential has two bumps of different height, the smaller one of which has a non-degenerate maximum of value E_0.

Let

$$\delta_0 := \frac{1}{2}\Big(\sum_{j=1}^{n} \lambda_j - \lambda_1 d\Big), \tag{1.2}$$

where again $0 < \lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of $\sqrt{-2V''(x_0)}$ and d is the dimension of the set of homoclinic trajectories, which should be at most n. The quantity δ_0 is positive since $\delta_0 = \frac{1}{2}\big(\sum_{j=1}^{n}(\lambda_j - \lambda_1) + \lambda_1(n-d)\big)$, and $\delta_0 = 0$ if and only if $\lambda_1 = \cdots = \lambda_n$ and $d = n$. Then we will show in Theorem 2.4, under additional assumptions, that *if the quantity δ_0 is strictly positive, then there exists $\delta > 0$ such that for any $C > 0$, there is no resonance in $(E_0 - Ch, E_0 + Ch) - i(0, \delta h)$ for sufficiently small h.* In fact, the optimal δ coincides probably with δ_0 (see Remark 4.2).

This theorem says that the trapped set can be of large dimension out of the fixed point. Instead, the smallness of the trapped set is required near the fixed point by the assumption $\delta_0 > 0$ (see Remark 2.3).

A simplest example where $\delta_0 = 0$ is the case $n = 1$. The precise asymptotic distribution of resonances in this case was studied in [9] and the width of resonances is of order $\frac{h}{|\log h|}$.

The rest of this paper is organized as follows. In the next section, we state the precise statement of the result, and in the third section, we give a sketch of the proof. It is essentially based on the connection formula at the hyperbolic fixed point proved in [4], which gives the WKB solution on the outgoing stable manifold in terms of that on the incoming one via a Fourier integral operator. A brief survey of this formula will be given in Appendix.

2. Result

In our result, we will need the analyticity of $V(x)$ only in the vicinity of ∞, i.e.

(A1) $V(x) \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and extends holomorphically in a sector

$$\widetilde{\mathcal{S}} = \{x \in \mathbb{C}^n;\ |\operatorname{Im} x| \leq \tan\theta_0 \langle \operatorname{Re} x\rangle,\ |\operatorname{Re} x| > C\},$$

for some positive constants θ_0 and C. Moreover $V(x)$ tends to 0 as x tends to ∞ in $\widetilde{\mathcal{S}}$.

In addition, we assume the following condition.

(A2) The origin is a non-degenerate maximal point with maximal value $E_0 > 0$, i.e. for a suitable choice of coordinates,

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(x^3) \quad \text{as } x \to 0, \tag{2.1}$$

for positive constants $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

Let $\xi \in \mathbb{R}^n$ denotes the momentum. Then **(A2)** means that the origin $(0,0)$ of the phase space $T^*\mathbb{R}^n = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is a hyperbolic fixed point for the Hamiltonian vector field H_p. The fundamental matrix F_p, which is the linearization of H_p at the origin, is

$$F_p = \begin{pmatrix} 0 & 2\mathrm{Id} \\ \frac{1}{2}\mathrm{diag}\,(\lambda_1^2, \dots, \lambda_n^2) & 0 \end{pmatrix},$$

and its eigenvalues are $\pm\lambda_1, \dots, \pm\lambda_n$.

Let Ω be a small neighborhood of $(0,0)$. By the stable/unstable manifold theorem, there exist the outgoing and incoming stable manifolds Λ_+, Λ_- associated to this fixed point:

$$\Lambda_\pm := \{(x,\xi) \in \Omega;\ \exp(tH_p)(x,\xi) \to (0,0) \quad \text{as}\, t \to \mp\infty\} \subset p^{-1}(E_0).$$

They are tangent to the planes $\{(x,\xi);\ \xi_j = \pm\lambda_j x_j/2,\ 1 \leq j \leq n\}$ respectively at $(0,0)$, and Lagrangian manifolds with generating functions $\phi_\pm(x)$ behaving like $\phi_\pm(x) = \pm\sum_{j=1}^n \frac{\lambda_j}{4} x_j^2 + \mathcal{O}(x^3)$ in a vicinity of 0, i.e.

$$\Lambda_\pm = \Lambda_{\phi_\pm} = \{(x,\xi) \in \Omega;\ \xi = \frac{\partial\phi_\pm}{\partial x}\}.$$

Suppose $(x_\pm, \xi_\pm) \in \Lambda_\pm$. By definition of $\Lambda_\pm$, the curve $\exp(tH_p)(x_\pm,\xi_\pm) \to (0,0)$ as $t \to \mp\infty$. More precisely, it was proved in [11] that they have asymptotic expansion:

$$\exp(tH_p)(x_\pm,\xi_\pm) \sim \sum_{k=1}^{\infty} \gamma_k^\pm(t, x_\pm, \xi_\pm) e^{\pm\mu_k t} \quad \text{as} \quad t \to \mp\infty,$$

where $0 < \mu_1 < \mu_2 < \cdots$ are the linear combinations over $\mathbb{N} = \{0, 1, \dots\}$ of $\{\lambda_j\}_{j=1}^n$, and in particular $\mu_1 = \lambda_1$. The coefficients $\gamma_k^\pm(t, x_\pm, \xi_\pm)$ are vector valued polynomials in t depending on the initial point $(x_\pm, \xi_\pm)$, and in particular γ_1 is an eigenvector of F_p corresponding to $\pm\lambda_1$, and it is independent of t. In our Schrödinger case, the x-space projection of $\gamma_1^+(x_+, \xi_+)$ and $\gamma_1^-(x_-, \xi_-)$ coincide if $x_+ = x_- = x$, and we will denote it simply by $g(x)$.

Next we assume that the trapped set consists of the fixed point and the associated homoclinic trajectories. Let $\tilde\Lambda_\pm$ be the evolution of $\Lambda_\pm$ by the Hamiltonian flow to large $\pm t$. Let us denote the set of homoclinic trajectories $\tilde\Lambda_+ \cap \tilde\Lambda_-$ by $\mathcal{N}$ and its x-space projection by $\mathcal{N}_x$. The third assumption is the following:

- **(A3)** $\mathcal{N}$ is a non-empty manifold of dimension $d\,(\leq n)$, and $K(E_0) = \{(0,0)\} \cup \mathcal{N}$. Moreover, $T_\rho\mathcal{N} = T_\rho\tilde\Lambda_+ \cap T_\rho\tilde\Lambda_-$ for any $\rho = (x,\xi) \in \mathcal{N}$.

Remark 2.1. *In fact, the manifold $\tilde\Lambda_+$ projects diffeomorphically on $\mathbb{R}^n_x$ near $\mathcal{N}$ in the vicinity of 0, and hence has a generating function $\tilde\phi_+(x)$, see* [1, Appendix C] *in the case $d = 1$. The latter assumption* **(A3)** *implies that:*

$$\mathrm{rank}\Big[\partial_x^2\big(\tilde\phi_+(y) - \phi_-(y)\big)\Big] = n - d, \quad \textit{for all } y \in \mathcal{N}_x. \tag{2.2}$$

We assume also that:

(A4) $g(y_1) \cdot g(y_2) \neq 0$ for $y_1, y_2 \in \mathcal{N}_x$.

Remark 2.2. *Let $q \in \{1, \dots, n\}$ be the maximal number satisfying $\lambda_1 = \cdots = \lambda_q$. Then* **(A4)** *implies that $\mathcal{N}_x$ is tangent to $\mathbb{R}^q_{x_1,\dots,x_q}$ at $(0,0)$ as $t \to +\infty$ and that any two vectors tangent to $\mathcal{N}_x$ at $(0,0)$ are not orthogonal to each other.*

As explained in the introduction (see (1.2)), we eventually assume that:

(A5) $\delta_0 := \frac{1}{2}\Big(\sum_{j=1}^n \lambda_j - \lambda_1 d \Big)$ is strictly positive.

Remark 2.3. *This condition is equivalent to either $\lambda_1 < \lambda_n$ or $d < n$. Geometrically, this means, under* **(A4)***, that the "angle" of $\mathcal{N}_x$ at $x = 0$ is 0,*

$$\lim_{r \to 0} \frac{|\mathcal{N}_x \cap \{|x| = r\}|}{r^{n-1}} = 0.$$

Then our result is the following:

Theorem 2.4. *Assume* **(A1)**, **(A2)**, **(A3)**, **(A4)** *and* **(A5)**. *Then there exists a constant δ with $0 < \delta \leq \delta_0$ such that for any $C > 0$, there exists $h_0 > 0$ such that, for any $h \in]0, h_0]$, P has no resonance in*

$$R_{\delta,h} :=]E_0 - Ch, E_0 + Ch[- i[0, \delta h[. \tag{2.3}$$

Moreover, for $\chi \in C_0^\infty(\mathbb{R}^n)$, there exist positive constants N, K and h_0 such that, for any E in $R_{\delta,h}$ and $0 < h < h_0$, one has

$$||\chi(P - E)^{-1}\chi|| \leq K h^{-N}.$$

3. Sketch of proof

It is enough to prove that $\|(P_\theta - E)^{-1}\| \leq K h^{-N}$ for $E \in R_{\delta,h}$. For this, we proceed by contradiction (as in [7] for the limiting absorption principle). If this did not hold, then there would exist $u = u(x,h)$ satisfying $||u|| = 1$ and

$$(P_\theta - E)u = \mathcal{O}(h^\infty). \tag{3.1}$$

We suppose that u satisfies (3.1) for $||u|| = 1$ and $E \in R_{\delta,h}$ with $\delta = \min(\delta_0, \delta_1)$ where δ_1 is given in Theorem 4.1 below. We will show that:

$$||u|| = \mathcal{O}(h^\infty). \tag{3.2}$$

Let us look at u microlocally in the phase space. We say that u is microlocally 0 at a point $\rho = (x, \xi)$ in $\mathbb{R}^{2n}_{x,\xi}$ if

$$||\mathrm{Op}_h(\psi)u|| = \mathcal{O}(h^\infty),$$

for some $\psi \in C_0^\infty(\mathbb{R}^{2n})$ with $\psi = 1$ near ρ, where $\mathrm{Op}_h(\psi)$ is an h-pseudodifferential operator with symbol ψ given by:

$$\mathrm{Op}_h(\psi)u = \frac{1}{(2\pi h)^n} \int\int e^{i(x-y)\cdot\xi/h} \psi(\frac{x+y}{2}, \xi) u(y) dy d\xi.$$

By the ellipticity, u is microlocally 0 outside the energy surface $p^{-1}(E_0)$. Moreover, the fact (3.1) with $||u|| \leq 1$ implies that u is microlocally 0 in the incoming region

$$\big\{(x,\xi) \in p^{-1}(E_0);\ |x| \gg 1,\ \ \cos(x,\xi) := \frac{x\cdot\xi}{|x||\xi|} < -\frac{1}{2}\big\},$$

see [6, Theorem 2]. Then, with the standard propagation of singularities, it turns out that u is microlocally 0 outside $\tilde{\Lambda}_+$, see for example [13]. In particular, on Λ_-, u is localized only on $\mathcal{N}$.

Let $u_\pm$ be the restriction of u on $\mathcal{N}\cap\Lambda_\pm$ respectively. We can relate these two microlocal solutions in two ways: from u_- to u_+ passing through the fixed point, and from u_+ to u_- following the homoclinic trajectories. For the first connection, we use the results in [4], which will be recalled briefly in Appendix, and for the second, we use the standard Maslov theory, see [8].

We first apply Theorems 4.1 and 4.4 of Appendix in order to obtain u_+ from u_-. It is possible because we took $\delta \leq \delta_1$ and the assumption **(H)** needed for Theorem 4.4 is guaranteed by **(A4)**.

By assumption, we have $||u_-|| \leq 1$. By a first use of those theorems, we see that u_+ is a Lagrangian distribution of order h^{-C} for some C,

$$u_+ \in I(\Lambda_+\backslash\{(0,0)\}, h^{-C}).$$

Next, using the Maslov's theory along $\tilde{\Omega}_+$ a small neighborhood of $\mathcal{N}$, we see that u is always Lagrangian of the same order:

$$u_- \in I(\tilde{\Lambda}_+, h^{-C}). \tag{3.3}$$

Then, by a second use of Theorem 4.4, we obtain

$$u_+ \in I(\Lambda_+\backslash\{(0,0)\}, h^{-C+\alpha}), \tag{3.4}$$

where $\alpha := (\delta_0 - |\operatorname{Im} z|)/\lambda_1$ and $E = E_0 + hz$. Let us check this estimate.

First, we look at the prefactor in power of h of the integral (4.2). Its absolute value is $h^{(\beta-|\operatorname{Im} z|)/\lambda_1}$ with $\beta := \frac{1}{2}\sum_{j=1}^{n}(\lambda_j - \lambda_1)$.

Next, we look at the integral part of (4.2). (3.3) implies that:

$$u_- = b(x;h)e^{i\tilde{\phi}_+(x)/h}, \tag{3.5}$$

where $\tilde{\phi}_+(x)$ is a generating function of $\tilde{\Lambda}_+$ near $\mathcal{N}\cap\Lambda_-$, see Remark 2.1. Then the integral part of (4.2) reads

$$e^{i\phi_+(x)/h}\int_{\mathbb{R}^{n-1}} e^{i(\tilde{\phi}_+(\varepsilon,y')-\phi_-(\varepsilon,y'))/h} d(x,y';h)b(\varepsilon,y';h)dy'. \tag{3.6}$$

Assumption **(A3)** means that the integral (3.6) has $(n-d)$ directions along which the phase $[\tilde{\phi}_+(\varepsilon,y') - \phi_-(\varepsilon,y')]$ has non-degenerate critical points on $\mathcal{N}$, see (2.2). Then, by the stationary phase, this integral is of order $h^{\frac{(n-d)}{2}}$.

Thus the order of the Lagrangian distribution u_+ is higher by $(\delta_0 - |\operatorname{Im} z|)/\lambda_1 =: \alpha$ than the order of u_-, and we get (3.4).

Since α is positive for $E \in R_{\delta,h}$, we conclude that u_- is $\mathcal{O}(h^\infty)$ by repeating the same argument. Again by Theorem 4.1, u is microlocally 0 on $\tilde{\Lambda}_+$, and hence everywhere in the phase space. This is equivalent to (3.2).

4. Appendix

This section is a short survey of the results in [4], which were used for the proof of Theorem 2.4.

We consider a Schrödinger operator P whose potential V is smooth near $x = 0$ with Taylor expansion (2.1) with $E_0 = 0$. We use the same notations as in Section 3.

For $\varepsilon > 0$ small, we consider the microlocal Cauchy problem, with $E = hz$,

$$\begin{cases} Pu = hzu & \text{in } \Omega, \\ u = u_0(x) & \text{on } \mathcal{C} := \Lambda_- \cap \{|x| = \varepsilon\}. \end{cases} \tag{4.1}$$

Remark that the initial surface $\mathcal{C}$ is transversal to the Hamiltonian vector field for sufficiently small ε.

Theorem 4.1. *There exists a constant $\delta_1 > 0$ such that if $u_0 = 0$ and if $z(h)$ is in $]-C, C[-i[0, \delta_1[$ for any constant $C > 0$, then the solution $u \in L^2(\mathbb{R}^n)$ of (4.1) with $\|u\| \leq 1$, is microlocally 0 in a neighborhood Ω' of the origin.*

Remark 4.2. *More precisely, Theorem 4.1 holds for $z(h)$ outside any small neighborhood of size h of some discrete set. On the other hand, Theorem 4.1 holds also in the analytic category, changing of course the notion of C^∞-microsupport to that of analytic microsupport and in this case the exceptional discrete set is known to be $-i\mathcal{E}_0$ where*

$$\mathcal{E}_0 = \Big\{ \sum_{j=1}^{n} \lambda_j(\alpha_j + \frac{1}{2});\ (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \Big\},$$

is the set of eigenvalues of the harmonic oscillator $-\Delta + \sum_{j=1}^n \frac{\lambda_j^2}{4} x_j^2$. In particular, δ_1 can be taken to be $\big(\frac{1}{2}\sum_{j=1}^n \lambda_j - a\big)$, for any small $a > 0$, which is greater than δ_0.

Theorem 4.1 says that the data u_0 given on $\Lambda_- \cap \{|x| = \varepsilon\}$ uniquely determines the solution u at any point $\rho_F = (x, \xi)$ on Λ_+ (if it exists). Next theorem enables us to represent u near ρ_F in terms of u_0 which, restricted to the initial surface $\mathcal{C}$, has its support in a small neighborhood of a point $\rho_I = (y, \eta) \in \mathcal{C}$.

We make an assumption on the initial point $\rho_I = (y, \eta) \in \mathcal{C}$ and the final point $\rho_F = (x, \xi) \in \Lambda_+$:

(H) $g(x) \cdot g(y) \neq 0$.

Remark 4.3. *Let $\phi_1(x)$ be the function defined from ρ_I by*

$$\begin{cases} 2\nabla\phi_+\cdot\nabla\phi_1-\lambda_1\phi_1=0,\\ \nabla\phi_1(0)=-\lambda_1 g(y).\end{cases}$$

Then **(H)** *implies $\phi_1(x)\neq 0$.*

We assume, without loss of generality, that $g(y)$ is parallel to x_1-axis. Since p is of real principal type near ρ_I, we can modify the initial surface $\mathcal{C}$ so that it is given by $\{x_1=\varepsilon\}\cap\Lambda_-$ near ρ_I. Hence, denoting $y=(\varepsilon,y')$, the initial data u_0 on $\mathcal{C}$ is a function of x' localized in a small neighborhood of x'_I.

Let $\psi(x)$ be the solution to the Cauchy problem of the eikonal equation:

$$\begin{cases} |\nabla\psi|^2+V(x)=0,\\ \\ \psi|_{x_1=\varepsilon}=\eta'\cdot x', \quad \text{where } \eta'=\frac{\partial\phi_-(\varepsilon,y')}{\partial y'}.\end{cases}$$

Then, $\psi(x)$ behaves like

$$\psi(x)=-\frac{\lambda_1}{4}x_1^2+\sum_{j=2}^{n}\frac{\lambda_j}{4}x_j^2+\mathcal{O}(x^3)\quad \text{as } x\to 0,$$

and the integrals

$$I_+^\infty(x):=\int_0^{-\infty}\big(\Delta\phi_+(x(\tau))-\nu\big)\,d\tau\quad\text{and}\quad I_-^\infty(y):=\int_0^{+\infty}\big(\Delta\psi(y(\tau))-\nu+\lambda_1\big)\,d\tau,$$

converge. Here $\nu:=\frac{1}{2}\sum_{k=1}^n\lambda_k$.

Theorem 4.4. *Assume $z\in]-C,C[-i[0,\delta_1[$ and* **(H)***. Then the microlocal Cauchy problem (4.1) has a unique solution u, and near $\rho_F=(x,\xi)$, it has the integral representation:*

$$u(x,h)=\frac{h^{S(z)}}{(2\pi h)^{n/2}}\int_{\mathbb{R}^{n-1}}e^{i(\phi_+(x)-\phi_-(\varepsilon,y'))/h}d(x,y';h)u_0(y')dy'. \tag{4.2}$$

Here $S(z)=\big(\frac{1}{2}\sum_{j=1}^n(\lambda_j-\lambda_1)-iz\big)/\lambda_1$ and the symbol $d\in S_h^0(1)$ has the asymptotic expansion

$$d(x,y';h)\sim\sum_{k=0}^{\infty}d_k(x,y',\ln h)h^{\hat\mu_k/\lambda_1}, \tag{4.3}$$

where $0=\hat\mu_0<\hat\mu_1(=\mu_2-\mu_1)<\hat\mu_2<\cdots$ is a numbering of the linear combinations of $\{\mu_k-\mu_1\}_{k=0}^\infty$ over $\mathbb{N}$, and $d_k(x,y',\ln h)$ are polynomials in $\ln h$. In particular, d_0 is independent of $\ln h$ and given by

$$\begin{aligned} d_0(x,y)=&e^{-i\frac{\pi n}{4}}\lambda_1^{\frac{1}{2}-S(z)}\exp\Big(-\frac{\pi}{2}i\,\sigma\,S(z)\Big)\Gamma\left(S(z)\right)\\ &\times e^{I_+^\infty(x)-I_-^\infty(y)}\sqrt{|\det\nabla_{y'}^2\phi_-(y)|}\cdot\frac{|g(y)|}{|g(x)\cdot g(y)|^{S(z)}}\ , \end{aligned}\tag{4.4}$$

where $\sigma=\operatorname{sgn}\big(g(x)\cdot g(y)\big)$. Here $\operatorname{sgn}(t)=\frac{t}{|t|}$ for all $t\in\mathbb{R}\setminus\{0\}$.

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