NON LINEAR EIGENVALUE PROBLEMS AND RELATED TOPICS

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Dedicated to Professor Yoshinori Morimoto's 60'th birthday

In this note, I would like to give a survey of recent developments of nonlinear eigenvalue problems, and explain my personal motivation to such subjects.

(1) **What is "nonlinear eigenvalue problem".**

Let $L(\lambda) = H_0 + \lambda H_1 \cdots + \lambda^{m-1} H_{m-1} + \lambda^m I$, where $H_j (j = 0, \ldots, m-1)$ are operators defined in some Hilbert space $\mathfrak{H}$ and λ is a complex parameter($L(\lambda)$ is called an operator pencil). The problem to study distribution of $\lambda \in \mathbf{C}$ where $L(\lambda)u = 0$ for some $0 \neq u \in \mathfrak{H}$ is called a nonlinear eigenvalue problem. In the simplest case $L(\lambda) = H + \lambda I$ we come to usual spectral problem. Nonlinear eigenvalue problems can be regarded as subjects in eigenvalue problems of non-selfadjoint operators. This is because $L(\lambda)$ is not invertible if and only if $\mathfrak{L} - \lambda I$ is not invertible, where

$$\mathfrak{L} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \ldots & 1 \\ -H_0 & -H_1 & -H_2 & \ldots & -H_{m-1} \end{pmatrix}.$$

There is a classical text book by Markus [6] and the theory of these problems had been studied by Russian mathematicians for long time.

In the motivations to study nonlinear eigenvalue problems, there is a problem on decay rate of the solutions to the damping wave equations. For example, let us consider the following problem:

$$\begin{cases} u_{tt} - u_{xx} + 2a(x)u_t = 0 & (x,t) \in]0,\,1[\times \mathbf{R}_+, \\ \quad u(0,t) = u(1,t) = 0, & t \in \mathbf{R}_+, \end{cases}$$

The author was supported in part by Grant-in-Aid for Scientific Research,Ministry of Education,Culture,Sports, Science and Technology, Japan (No.22654019).

The function $a(x)$ is called the damping term. By using Fourier transform with respect to vairable t, the above problem becomes a nonlinear eigenvalue problem of the quadratic operator pencil as follows :$-y_{xx} + 2\lambda a(x)\, y + \lambda^2 y = 0$. In the case where $a(x) \equiv a_0$ is a positive constant, one can easily see that the term a_0 shifts the eigenvalues to the left half plane $\mathrm{Re}\lambda < 0$ and this implies the (exponential) decay of the solutions.

(2) **Nonlinear eigenvalue problems and non-analytic hypoellipticity.**

The relation between non-analytic hypoellipticity and nonlinear eigenvalue problems was found by Pham The Lai and Robert [7]. They solved a question proposed by B. Helffer as follows : can we show the existence of (λ, u) where $\lambda \in \mathbf{C}$ and $u \neq 0$ in Schwartz space satisfying

$$-u_{xx} + (x^2 - \lambda)^2 u = 0, \quad x \in \mathbf{R}.$$

Clearly this is a nonlinear eigenvalue problem($H_0 = -(\partial_x)^2 + x^4$ and $H_1 = 2x^2$). The existence implies the existence of non-analytic solution of the hypoelliptic operator $D_x^2 + (D_y - x^2 D_t)^2$, which is very important because the similar operator $D_x^2 + (D_y - xD_t)^2$ is known to be analytic hypoelliptic.

(3) **My career and motivation.**

I had started mathematical career by studying hypoellipticity of infinitely degenerate elliptic operators. I was very influenced by the important works of Professor Yoshinori Morimoto in mid 80's. Let me explain some history. After the pioneering result on infinitely degenerate operators by Fedii, Professor Morimoto made the generalization of Fedii's work. However we were surprised when we knew the work of Kusuoka and Stroock in 1985. They proved that the operator $D_x^2 + \exp(-\frac{1}{|x|^\sigma})D_y^2 + D_t^2$ $(\sigma > 0)$ is hypoelliptic if and only if $\sigma < 1$. This fact is similar to Bouendi-Goulaouic's example: the operator $D_x^2 + x^2 D_y^2 + D_t^2$ is hypoelliptic but not analytic hypoelliptic. Professor Morimoto found microlocal proof of their result and made its generalization. After Professor Morimoto's works, I proved that the operator $D_x^2 + \exp(-\frac{1}{|x|^\sigma})D_y^2 + x^{2k} D_t^2$ is hypoelliptic if and only if $\sigma < k + 1$. This is similar to the following fact (due to Professor Okaji): the operator $D_x^2 + x^{2k} D_y^2 + x^{2\ell} D_t^2$ is analytic hypoelliptic if and only if $k = \ell$. Thus concerning the study of hypoellipticity of infinitely degenerate elliptic operators, one should take care of the relation to non-analytic hypoellipticty of the similar operators.

In the first 90's,Michael Christ [3] had succeeded to generalize the above work of Pham The Lai and Robert [7]. He proved the existence of (λ, u) where $\lambda \in \mathbf{C}$ and $u \neq 0$ in Schwartz space satisfying

$$-u_{xx} + (x^m - \lambda)^2 u = 0, \quad x \in \mathbf{R},$$

for positive integers m satisfying $m > 1$. His result implies non-analytic hypoellipticity of the operator $D_x^2 + (D_y - x^m D_t)^2$. This result motivated me to consider the hypoellipticity of the operators of the form $D_x^2 + (f(x)D_y - g(x)D_t)^2$. As a preparation to this problem, I proved that the operator $D_x^2 + (x^k D_y - x^\ell D_t)^2$ $(k < \ell)$ is non-analytic hypoelliptic if either of the cases where (i) $(\ell+1)/(\ell-k)$ is not integer ,or (ii) the both $\ell - k$ and $(\ell+1)/(\ell-k)$ are odd integers([5]). However the my research had not developped in this area, and I changed my interest to new subject, namely , the smoothing effect of dispersive equations. Lastly I have received a preprint from Costins'([4]) at Rutgers University(2000). They succeeded to prove my remaining cases, namely they proved that the operator $D_x^2 + (x^k D_y - x^\ell D_t)^2$ is not analytic hypoelliptic except the case $(k, \ell) = (0, 1)$. But I could not come back to these subjects.

(4) **Recent Developments on nonlinear eigenvalue problems.**

In September 2009, I attended the colloquium at Orsay(which celebrated Helffer's 60'th birth day). There was Professor Chanillo as a speaker. He belongs to Rutger's University as Contin's and Treves. He knew my name, and after greeting him, I became to know that some progresses have been made . Here is an explanation of these progresses.

(i) **New approach by Chanillo-Helffer-Laptev.**

The methods of Pham The Lai-Robert and Christ rely on some estimates of the wronskians with respect to the spectral parameter. So it seems difficult to apply their methods to the operators in higher dimensions. Chanillo, Helffer and Laptev [2] gave a new approach which is so general to apply many operators. Their method deals with trace of the operator pencils and uses Lidskii's theorem. The following is outline of the proof of M. Christ's result by their method.

First let us write the operator $D_x^2 + (x^m - \lambda)^2$ by $H = L - 2\lambda M + \lambda^2$ where $L = D_x^2 + x^{2m}$, $M = x^m$ and put $L^{-1/2} H L^{-1/2} = I - 2\lambda B + \lambda^2 A$. Then the operator $D_x^2 + (x^m - \lambda)^2$ is not invertible if and only if $I - \lambda \mathfrak{C}$

is not invertible, where

$$\mathfrak{C} = \begin{pmatrix} 2B & A^{1/2} \\ -A^{1/2} & 0 \end{pmatrix}.$$

Our remaining task is to show that the operator $\mathfrak{C}$ has an eigenvalue $\mu \neq 0$, because then $D_x^2 + (x^m - \lambda)^2$ is not invertible if we take $\lambda = \mu^{-1}$.

Lidskii's theorem says that if the operator $\mathfrak{D}$ is of trace class then $\mathrm{Tr}\ \mathfrak{D} = \sum \mu_j(\mathfrak{D})$, where $\mu_j(\mathfrak{D})$ are the eigenvalues of $\mathfrak{D}$ (the definition of "trace" is $\mathrm{Tr}\ \mathfrak{D} = \sum(\mathfrak{D}e_k, e_k)$ for some orthonormal basis $\{e_k\}$). Hence one can see that the operator $\mathfrak{D}$ has a non-zero eigenvalue if $\mathrm{Tr}\ \mathfrak{D} \neq 0$. The above operator $\mathfrak{C}$ is not of trace class, but from the distribution of the eigenvalues of L , we can see that A is of trace class and B is a Hilbert-Schmidt operator, hence the operator $\mathfrak{C}^2$ is of trace class. Thus, our remaining task is to show that $\mathrm{Tr}(\ \mathfrak{C}^2) = \mathrm{Tr}(2B^2 - A) \neq 0$.

The key of calculation of the trace is the fact that $D_x^2 + \gamma x^{2m}$ is isospectral to $\gamma^{-1/(m+1)}(D_x^2 + x^{2m})$. One can see this fact by a scaling transformation. Thus it holds that $\mathrm{Tr}((D_x^2 + \gamma x^{2m})^{-1}) = \mathrm{Tr}((\gamma^{-1/(m+1)}(D_x^2 + x^{2m}))^{-1})$ and furthermore by differentiation this equality with respect to γ and taking $\gamma = 1$, we obtain $\frac{1}{m+1}\mathrm{Tr}(A) = \mathrm{Tr}(Ax^{2m}A) = \mathrm{Tr}(B^2)$. Thus

$$\mathrm{Tr}(2B^2 - A) = \left(\frac{2}{m+1} - 1\right)\mathrm{Tr}(A) < 0$$

if $m > 1$, which induces the result of M. Christ.

Remark: The above method is not effective for the operator $D_x^2 + (x^\ell - x^k\lambda)^2$ $(k < \ell)$. It can be shown the existence of the eigenvalues under the strong assumption $2k + 1 < \ell$.

(ii) **Progress by Robert.**

The method of Chanillo-Helffer-Laptev is very general to apply the operators in higher dimensions. However the assumptions in their paper becomes too restrictive in general. For example, concerning the operators of the form $-\Delta + (P(x) - \lambda)^2$ ($P(x)$: elliptic homogeneous polynomial) in dimension $n = 2, 3$, they suppose that the degree of the polynomial $P(x)$ is greater than 6. This is because their derivation of the trace is ad hoc. After the novel work by Chanillo-Helffer-Laptev, Robert applied the semi-classical pseudodifferential operator calculus to obtain some information of the trace for an operator related to $-\Delta + (P(x) - \lambda)^2$. Here I would like to sketch the calculation of the trace.

For simplicity, we suppose here that the polynomial $P(x)$ is elliptic and positively homogeneous of degree $m \geq 2$. At first note that by the scaling transformation $x = \tau^{1/m} y$ with $\hbar = \tau^{-(m+1)/m}$ and $z = \lambda/\tau$, the operator $L(\lambda) = -\Delta + (P(x) - \lambda)^2$ is unitary equivalent to $\tau^2 \hat{L}(z)$, where $\hat{L}(z) = -\hbar^2 \Delta + (P(x) - z)^2$ is the $\hbar-$ Weyl operator with symbol $L(z, x, \xi) = |\xi|^2 + (P(x) - z)^2$, and we can construct its parametrix. The symbol of leading term of the parametrix is $K(z, x, \xi) = \frac{1}{L(z,x,\xi)} = \frac{1}{|\xi|^2 + (P(x) - z)^2}$, which has poles at $z = P(x) \pm i|\xi|$.

On the other hand, let

$$\mathfrak{L} = \begin{pmatrix} 0 & I \\ \hbar^2 \Delta - P(x)^2 & 2P(x) \end{pmatrix}.$$

Then the trace formula says that

$$\mathrm{Tr}(f(\mathfrak{L})) = \sum_{\lambda \in \sigma(\mathfrak{L})} m_\lambda f(\lambda) = \mathrm{Tr}\left(\frac{1}{2\pi i} \int_\Gamma \hat{L}(z)^{-1} \hat{L}(z)' f(z)\, dz \right)$$

for an appropriate contour Γ and $f(\lambda)$ sufficiently decaying to infinity. The trace has the semi-classsical expansion $\mathrm{Tr}(f(\mathfrak{L})) = \sum_{j \geq 0} C_{2j}(f) \hbar^{2j-n}$, and the leading term $C_0(f)$ comes from the leading term of the parametrix. More precisely,

$$\begin{aligned} C_0(f) &= \frac{1}{2\pi i} \left(\frac{1}{2\pi} \right)^n \int_\Gamma \iint_{\mathbf{R}^n \times \mathbf{R}^n} \frac{2(P(x) - z)}{|\xi|^2 + (P(x) - z)^2} dx d\xi\, f(z) dz \\ &= \left(\frac{1}{2\pi} \right)^n \iint_{\mathbf{R}^n \times \mathbf{R}^n} [f(P(x) + i|\xi|) + f(P(x) - i|\xi|)] dx d\xi \\ &= (i^n + (-i)^n) \left(\frac{1}{2\pi} \right)^n \iint_{\mathbf{R}^n \times \mathbf{R}^n} f(P(x) + |\xi|) dx d\xi \end{aligned}$$

Hence if the space dimension n is even, one can see that the leading term $C_0(f) \neq 0$ for $f(\lambda) = (1 + \lambda^2)^{-m}$ (where m is taken sufficiently large), and therefore the operator $\mathfrak{L}$ has non-zero eigenvalues. In the case where the space dimension n is odd, $C_0(f)$ becomes zero, but some consideration of $C_2(f)$ or $C_4(f)$ gives partial conclusions.

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