The Lyapunov exponents for nonhomogeneous linear differential systems

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In this note based on joint works with my graduate student LEE Hyung-Ju, we are concerned with the asymptotic behaviour of solutions to the following systems of linear differential equations with variable coefficients

$$\frac{dx}{dt} = A(t)x(t), \tag{1}$$

$$\frac{dy}{dt} = A(t)y(t) + f(t) \quad \text{in} \quad [t_0, \infty), \tag{2}$$

where $A(t)$ is a square matrix of order n whose components are bounded continuous on $[t_0, \infty)$. While $f(t)$, $x(t)$ and $y(t)$ are vector-valued functions of the classes C^0, C^1, C^1 on $[t_0, \infty)$ respectively.

First of all, let us recall the case of constant coefficients, that is, A is independent of t. Then we know the precise asymptotics for the solutions of (1) and (2) by the spectral resolution of e^{tA} like

$$e^{tA} = \sum_{\lambda \in \sigma(A)} e^{\lambda t} \sum_{j=0}^{h(\lambda)-1} \frac{t^j}{j!}(A - \lambda E)^j P_\lambda,$$

where $\sigma(A)$ is the spectrum of A, P_λ is the projection onto the generalized eigen-space $G(\lambda, h(\lambda)) = \{\ x \mid (A - \lambda E)^{h(\lambda)} x = 0\ \}$ and $h(\lambda)$ is the multiplicity of the eigenvalue λ which is the natural number satisfying $\operatorname{rank}(A - \lambda E)^{h(\lambda)+1} = \operatorname{rank}(A - \lambda E)^{h(\lambda)}$. Also, for initial data $x(t_0) = w$ define its degree $d_w(\lambda)$ by

$$d_w(\lambda) = \begin{cases} 0 & \text{if } P_\lambda w = 0, \\ k & \text{if } (A - \lambda E)^{k-1} P_\lambda w \neq 0,\ (A - \lambda E)^k P_\lambda w = 0. \end{cases}$$

Roughly speaking, we see that for any solution $x(t)$ of (1) with initial data $x(t_0) = w$

$$\|x(t)\| = \text{const. } e^{\alpha t} t^\beta + o\left(e^{\alpha t} t^\beta\right)$$

as $t \to \infty$, where $\alpha = \max_{\lambda \in \sigma(A)} \operatorname{Re} \lambda$, $\beta = \max_{\lambda \in \sigma(A)} d_w(\lambda)$. For (2) or the details, see Ishida-Lee [3]. So, it is natural to introduce the Lyapunov exponent of a vector-valued function $u : [t_0, \infty) \to \mathbb{C}^n$ as

$$\lambda(u) = \limsup_{t \to \infty} \frac{\log \|u(t)\|}{t}$$

due to Perron [6]. For instance, $\lambda\left(e^{at}t^b\right) = a$ when a, b are constants.

Remark 1. There is also an exponent ρ such that $\rho\left(e^{at}t^b\right) = b$, of course. See Ishida-Lee [3].

Question 1 How about the case that $A(t)$ converges to some const. A_0 as $t \to \infty$?

Our alternative interest is the case of diagonal $A(t) = \operatorname{diag}(a_{11}(t), \cdots, a_{nn}(t))$ because then (1) is reduced to n scalar equations $x_j'(t) = a_{jj}(t)x_j(t)$ with Lyapunov exponents

$$\lambda(x_j) = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\, ds. \tag{3}$$

Question 2 How about the case that $\lim_{t\to\infty} a_{jk}(t) = 0$ $(j \neq k)$? Then does (3) also hold?

Let us give a few affirmative answers to these questions under additional conditions below. The first milestone to Question 2 is achieved in Perron [6]. We impose that $\operatorname{Re} a_{11}(t) \geqq \operatorname{Re} a_{22}(t) \geqq \cdots \geqq \operatorname{Re} a_{nn}(t)$ for large t through Theorem 3.

Theorem 1 (Satz 7 in Perron [6], p. 765). Assume that the two conditions

$$\lim_{t\to\infty} a_{jk}(t) = 0 \quad \text{if} \quad j \neq k, \tag{4}$$

$$\liminf_{t\to\infty} \left(\operatorname{Re} a_{jj}(t) - \operatorname{Re} a_{j+1,j+1}(t)\right) > 0. \tag{5}$$

Then there exist n linearly independent solutions $x^1(t), \cdots, x^n(t)$ of (1) satisfying

$$\lambda(x^j) = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\, ds. \tag{6}$$

Question 3 Is the condition (5) essential? Namely, can we weaken it?

An answer to Question 3 is revealed by

Theorem 2 (Corollary 2 in Wang-Mai [7], p. 903). Let $a^\sharp(t) = \max_{j\neq k} |a_{jk}(t)|$. Assume that the two conditions

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} a^\sharp(s)\, ds = 0, \tag{7}$$

$$\operatorname{Re} a_{jj}(t) - \operatorname{Re} a_{j+1,j+1}(t) \geqq 2en\, a^\sharp(t) \ \text{ for large } t. \tag{8}$$

Then there exist n linearly independent solutions $x^1(t), \cdots, x^n(t)$ of (1) satisfying (6).

Corollary 1. If both $\lim_{t\to\infty} a^\sharp(t) = 0$ and (8) are fulfilled, then the same conclusion as in Theorem 1 holds.

Example 1 (Example 2 in Wang-Mai [7], p. 903). Let $n = 2$, $\alpha > 0$ and

$$A(t) = \begin{bmatrix} 1 + 12\, t^{-\alpha} & t^{-\alpha} \\ t^{-\alpha} & 1 \end{bmatrix}.$$

Then $\lambda(x^1) = 1$ and $\lambda(x^2) = 1$.

Question 4 What about the nonhomogeneous system (2)?

Theorem 3 (Lee [4]). Let $f^\sharp(t) = \max_{j=1,2,\cdots,n} |f_j(t)|$. Assume that (7),

$$\int^{\infty} f^\sharp(t)\,dt < \infty, \tag{9}$$

$$\operatorname{Re} a_{jj}(t) - \operatorname{Re} a_{j+1,j+1}(t) \geqq 2e\left(na^\sharp(t) + f^\sharp(t)\right), \tag{10}$$

$$\operatorname{Re} a_{jj}(t) \geqq a^\sharp(t) + e\, f^\sharp(t) \tag{11}$$

for large t. Then there exist n particular solutions $y^1(t), \cdots, y^n(t)$ of (2) satisfying

$$\lambda(y^j) = \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\,ds.$$

Remark 2. (10) corresponds to (8). But Theorem 3 does not contain Theorem 2 since the extra condition $\operatorname{Re} a_{jj}(t) \geqq a^\sharp(t)$ is unnecessary in Theorem 2. We note that (11) is technical. Further, Theorems 2, 3 are mere Corollaries of more general results in [7], [4].

What is the more general result in [4]? To state it, we need to prepare a few notions.

Definition 1. Let $r > 0$. We say that the nonhomogeneous system (2) is (n, r)-diagonal, if there are some $c > 0$ and $\varepsilon > 0$ such that for every $j = 1, 2, \cdots, n$

$$\int_{t_1}^{t_2} \left[\operatorname{Re} a_{jj}(t) - \frac{n}{r}a^\sharp(t) - \frac{1+\varepsilon}{r}\left(1 + \frac{1}{r}\right)^{j-1} f^\sharp(t)\right] dt \geqq -c$$

holds for any t_1, t_2 with $t_0 \leqq t_1 < t_2 < \infty$.

Definition 2. Let $N > 0$ and $a_j(t) = \max_{k=j+1,\cdots,n} \operatorname{Re} a_{kk}(t)$. We say that the nonhomogeneous system (2) has (n, N, r)-diversity, if every $j = 1, 2, \cdots, n$

$$\int_{t_1}^{t_2} \left\{\operatorname{Re} a_{jj}(t) - a_j(t) - \left[\left(N + n - 2 + \frac{n}{r}\right) a^\sharp(t) + \frac{N+1}{r} f^\sharp(t)\right]\left(1 + \frac{1}{r}\right)^{j-1}\right\} dt \geqq \log\frac{r}{N}$$

is satisfied for any t_1, t_2 with $t_0 \leqq t_1 < t_2 < \infty$.

"(n, r)-diagonality" is initially introduced in Lee [4], which ensures the non-vanishing property of some solutions to the nonhomogeneous system (2). Meanwhile, "(n, N, r)-diversity" is an evident extension of the similar one in [7] to the nonhomogeneous case.

Theorem 4 (Lee [4]). Assume that the nonhomogeneous system (2) is (n, r)-diagonal and that (2) has (n, N, r)-diversity for some N, r with $N \geqq r \geqq n + (1/2)$. Moreover, if

$$\int_{t_0}^{\infty} f^\sharp(t)\,dt < \infty, \tag{12}$$

then there exist n particular solutions $y^1(t), \cdots, y^n(t)$ of (2) whose Lyapunov exponents $\lambda_1, \cdots, \lambda_n$ fulfill the estimates

$$\left|\lambda_j - \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\,ds\right| \leqq \frac{n-1}{r + (1/2) - n}\ \limsup_{t\to\infty}\frac{1}{t}\int_{t_0}^{t} a^\sharp(s)\,ds.$$

Theorem 4 implies Theorem 3. Indeed, choose $\varepsilon = 1$ and $N = r = n + (1/2)$ in Theorem 4 and use the elementary inequalities

$$\left(1+\frac{1}{r}\right)^{j-1} < \left(1+\frac{1}{n}\right)^{n-1} < e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Next we shall answer Question 1 for the nonhomogeneous system (2). That is, assume that there exists a constant matrix A_0 such that $\lim_{t\to\infty} A(t) = A_0$. Let us denote $\sigma(A_0)$ by

$$\sigma(A_0) = \{\ \lambda_1,\ \lambda_2,\ \cdots,\ \lambda_r\ \},\quad \operatorname{Re}\lambda_1 > \operatorname{Re}\lambda_2 > \cdots > \operatorname{Re}\lambda_r,$$
$$\overbrace{\lambda_1,\cdots,\lambda_1}^{h(\lambda_1)\text{ in num.}},\ \overbrace{\lambda_2,\cdots,\lambda_2}^{h(\lambda_2)\text{ in num.}},\ \cdots,\ \overbrace{\lambda_r,\cdots,\lambda_r}^{h(\lambda_r)\text{ in num.}} =: \mu_1,\mu_2,\cdots,\mu_n.$$

Proposition 1. There is a fundamental matrix $X(t)$ to the homogeneous system (1) such that

$$X(t) = [x_1(t),\cdots,x_n(t)],\quad \lambda(x_k) = \operatorname{Re}\mu_k.$$

Proposition 2. $X^{-1}(x)^*$ is a fundamental matrix to the adjoint system $\dfrac{dz}{dt} = -A^*(t)z$.

Proposition 3 (Perron). Let $X^{-1}(t) =: [\eta_1(t),\cdots,\eta_n(t)]$ for $X(t)$ in Proposition 1. Then $\lambda(\eta_k) = -\operatorname{Re}\mu_k$.

These propositions can be found in [1] or [5]. Propositions 1, 3 are entirely non-trivial. The followings are some of our partial solution to Question 1 in aid of them.

Proposition 4. $\lambda(y) \leqq \max\{\operatorname{Re}\lambda_1, \lambda(f)\}$ for any solution y of (2).

We may actually construct solutions $y^1,\cdots,y^r$ of (2) such that $\lambda(y^k) = \operatorname{Re}\lambda_k$ for $k = 1,\cdots,r$ under the condition $\lambda(f) < \operatorname{Re}\lambda_r$.

Proposition 5. If there exists some $k \in \{1,\cdots,r\}$ with $\lambda(f) < \operatorname{Re}\lambda_k$, then there is a solution y of (2) satisfying $\lambda(y) < \operatorname{Re}\lambda_k$.

$\because$ If there is $\ell \in \{1,\cdots,n-1\}$ such that $\lambda(x_\ell) = \operatorname{Re}\lambda_k$ and $\lambda(x_{\ell+1}) < \operatorname{Re}\lambda_k$, then we put $\Phi^+(t) = {}^t[\eta_1(t),\cdots,\eta_\ell(t),0,\cdots,0]$ and $\Phi^-(t) = {}^t[0,\cdots,0,\eta_{\ell+1}(t),\cdots,\eta_n(t)]$, so that

$$y(t) = \int_{t_0}^{t} X(t)\Phi^-(s)f(s)\,ds + \int_{\infty}^{t} X(t)\Phi^+(s)f(s)\,ds$$

is the required one. Otherwise, take $\ell = n$, *i.e.*, $\Phi^+(t) = X^{-1}(t)$ and $\Phi^-(t) \equiv 0$. $\square$

Corollary 2. If there exists some $k \in \{1,\cdots,r\}$ with $\lambda(f) < \operatorname{Re}\lambda_k$, then there are solutions $y^1,\cdots,y^k$ of (2) satisfying $\lambda(y^j) = \operatorname{Re}\lambda_j$ for $j \in \{1,\cdots,k\}$.

Corollary 3. If $\lambda(f) < \operatorname{Re}\lambda_r$, then there are solutions $y^1,\cdots,y^r$ of (2) satisfying $\lambda(y^k) = \operatorname{Re}\lambda_k$ for $k \in \{1,\cdots,r\}$.

Remark 3. We never assume some condition on the rate of $A(t)$ to A_0 and the simplicity of $\sigma(A_0)$ bacause we do not employ any projection associated with A_0 which is represented by Dunford integral as usual in, *e.g.*, §3, Chap. IV in Coppel [2].

Remark 4. Recently, Lee in [5] has succeeded in removing the conditions (12), (11) thanks to the "regularity" of the homogeneous system (1) in Lyapunov's sense (See [1] or [5]).

Theorem 5 (Lee [5]). Assume that the two conditions of Wang-Mai [7]

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} a^\sharp(s)\, ds = 0, \tag{7}$$

$$\operatorname{Re} a_{jj}(t) - \operatorname{Re} a_{j+1,j+1}(t) \geqq 2en\, a^\sharp(t) \ \text{ for large } t \tag{8}$$

and that the limit

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\, ds$$

exists for every $j \in \{1, \cdots, n\}$. If

$$\lambda(f) < \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} a_{nn}(s)\, ds, \tag{13}$$

then there are n particular solutions $y^1(t), \cdots, y^n(t)$ of (2) fulfilling

$$\lambda(y^j) = \lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \operatorname{Re} a_{jj}(s)\, ds$$

for every $j \in \{1, \cdots, n\}$.

Remark 5. The condition (13) can be applied to several cases that $\int^{\infty} f^\sharp(t)\, dt = \infty$.

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