Construction of the fundamental solutions and spectral functions of nilmanilolds

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Let $F_{(N+N(N-1)/2)} \cong \mathbb{R}^N \oplus \mathbb{R}^{N(N-1)/2}$ be a connected and simply connected free 2-step nilpotent Lie group with the Lie algebra $\mathfrak{f}_{(N+N(N-1)/2)}$ (which also is identified with $\mathbb{R}^N \oplus \mathbb{R}^{N(N-1)/2}$). Fix a basis $\{X_i,\, Z_{i,j} \mid 1 \leq i,\, j \leq N,\, i < j\}$ of the Lie algebra $\mathfrak{f}_{(N+N(N-1)/2)}$.

Their bracket relation are assumed to be

$$[X_i, X_j] = 2Z_{ij} \quad 1 \leq i < j \leq N.$$

The group multiplication

$* : F_{(N+\ell)} \times F_{(N+\ell)} \to F_{(N+\ell)}$ ($\ell = N(N-1)/2$) is given by

$$\begin{aligned}\Big(\sum x_i X_i \oplus \sum z_{ij} Z_{ij}\Big) * \Big(\sum \tilde{x}_i\Big) X_i \oplus \sum \tilde{z}_{ij} Z_{ij}\Big) \\ = \sum \big(x_i + \tilde{x}_i\big) X_i \oplus \sum \big(z_{ij} + \tilde{z}_{ij} + x_i \tilde{x}_j - x_j \tilde{x}_i\big) Z_{ij}.\end{aligned}$$

Let $\tilde{X}_i$ be the left invariant vector fields on $F_{(N+\ell)}$ corresponds to X_i. We will construct the fundamental solution to the following degenerate parabolic equations on $F_{(N+\ell)}$:

$$\frac{d}{dt} e^w(t; x, D) + P e^w(t; x, D) = 0 \quad in \quad (0, T) \times \mathbf{R^n},$$
$$e^w(0; x, D) = I,$$

where

$$P = -\frac{1}{2} \sum_{i=1}^{N} \tilde{X}_i^2.$$

We can construct the fundamental solution $e^w(t; x, D)$ as pseudo-differential operator with parameter t according to C.Iwasaki and N.Iwasaki[3]. In our case the precise symbol $e(t; x, \xi)$

is obtained. By this formula we get the following theorem for the trace of the heat kernel $K_{L\backslash F_{(3+3)}}(t)$ on a nilmanifold $L\backslash F_{(3+3)}$, where $L$is a lattice defined by

$$L = \{ \ (m_1, m_2, m_3, k_1, k_2, k_3) \mid m_i, k_i \in \mathbb{Z} \ \}.$$

Theorem The heat kernel trace of $\Delta^{sub}_{L\backslash F_{(3+3)}}$ has the following small time asymptotic expansion:

$$K_{L\backslash F_{(3+3)}}(t) = \frac{\sqrt{\pi}}{32\sqrt{2}} t^{-9/2} + O(t^{\infty}).$$

We obtain the following theorem for the spectral zeta function:

$$\zeta^{sub}_{L\backslash F_{(3+3)}}(\ s\ ) = \frac{1}{\Gamma(\ s\ )} \int_0^{\infty} \left\{ \ K_{L\backslash F_{(3+3)}}(\ t\ ) - 1 \ \right\} t^{s-1} \, dt.$$

Theorem $\zeta^{sub}_{L\backslash F_{(3+3)}}(\ s\ )$ is meromorphic on the complex plane with one simple pole in $s = \frac{9}{2}$ and residue:

$$\mathrm{Res}_{s=\frac{9}{2}} \zeta^{sub}_{L\backslash F_{(3+3)}}(\ s\ ) = \frac{\sqrt{\pi}}{32\sqrt{2}\Gamma(\ \frac{9}{2}\ )} = \frac{1}{210\sqrt{2}}.$$

In particular, it follows that the spectral zeta-function is complex analytic in a zero-neighbourhood.

We have the similar results for $F_{(N+\ell)}$, where $\ell = N(N+1)/2$.

Theorem (1) The heat kernel trace of $\Delta^{sub}_{L\backslash F_{(N+\ell)}}$ has the following small time asymptotic expansion:

$$K_{L\backslash F_{(N+\ell)}}(t) = (2\pi t)^{-N/2-\ell} \int_{\mathbf{R}^{\ell}} W(\zeta) d\zeta + O(t^{\infty}).$$

(2) The spectral zeta function $\zeta^{sub}_{L\backslash F_{(N+\ell)}}(\ s\ )$ is meromorphic on the complex plane with one simple pole at $s = \frac{N}{2} + \ell$ and residue:

$$Res_{s=N/2+\ell} \zeta^{sub}_{L\backslash F_{(N+\ell)}}(\ s\ ) = \frac{1}{(2\pi)^{N/2+\ell}\Gamma(N/2+\ell)} \int_{\mathbf{R}^{\ell}} W(\zeta) d\zeta,$$

where $\Omega(\zeta)$ is a $N \times N$ skew symmetric matrix defined by

$$(\Omega(\zeta))_{ij} = \zeta_{ij} \quad (1 \le i < j \le N) \quad \text{and} \quad W(\zeta) = \sqrt{\det\left\{ \frac{\sqrt{-1}\Omega(\zeta)}{\sinh(\sqrt{-1}\Omega(\zeta))} \right\}}.$$

If we take

$$P = -\frac{1}{2} \sum_{i=1}^{N} \tilde{X}_i^2 - \frac{1}{2} \sum_{1 \le i < j \le N} \tilde{Z}_{ij}^2$$

instead of $P = -\frac{1}{2}\sum_{i=1}^{N}\tilde{X}_i^2$, the trace of the heat kernel $\Delta_{L\backslash F_{(N+\ell)}}$ and $\zeta_{L\backslash F_{(N+\ell)}}(\ s\ )$ have many singularities. For example, $\zeta_{L\backslash F_{(3+3)}}(\ s\ )$ has poles at $s = 1, 2, 3$ and their residues are

$$\mathrm{Res}_{|s=3-j}\zeta_{L\backslash F_{(3+3)}}(s) = \frac{2^{3j-2}}{(2j)!\pi^{\frac{7}{2}}} B_{2j}\left(\frac{1}{2}\right)\frac{\Gamma\left(j+\frac{3}{2}\right)}{\Gamma(3-j)} \quad (j = 0, 1, 2),$$

where $B_j(x)$ the j-th Bernoulli polynomial.

References

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