On fundamental solutions of diffusion equations related to non-local Dirichlet forms with BMO convections

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1 Introduction and Main Results

In this note we consider the non-local diffusion equations with a convective term

$$\partial_t\theta + A_K(t)\theta + v\cdot\nabla\theta = 0, \qquad t>0, \quad x\in\mathbb{R}^d, \tag{1.1}$$

where $d\ge 2$ and $A_K(t)$ is a linear operator formally defined by

$$\big(A_K(t)f\big)(x) = P.V.\int_{\mathbb{R}^d}(f(x)-f(y))K(t,x,y)\,\mathrm{d}y. \tag{1.2}$$

Here $K(t,x,y)$ is a positive function satisfying

$$K(t,x,y)=K(t,y,x), \qquad C_0^{-1}|x-y|^{-d-\alpha}\le K(t,x,y)\le C_0|x-y|^{-d-\alpha}, \tag{1.3}$$

for some constants $\alpha\in[1,2)$ and $C_0\ge 1$, and $v(t,x)=(v_1(t,x),\cdots,v_n(t,x))$ is a vector field satisfying the divergence free condition. The typical example of $A_K(t)$ is the fractional Laplacian $(-\Delta)^{\alpha/2}$ with $\alpha\in[1,2)$, which is defined by

$$(-\Delta)^{\frac{\alpha}{2}}f = C_{d,\alpha}\ \ P.V.\int_{\mathbb{R}^d}(f(x)-f(y))|x-y|^{-d-\alpha}\,\mathrm{d}y, \tag{1.4}$$

where $C_{d,\alpha}$ is a positive constant depending only on d and α.

The equations of the form (1.1) appear several models in the fluid dynamics, where the quantities are convected by the incompressible flows. In the case $d=2$ and $A_K(t)=(-\Delta)^{\alpha/2}$, if v and θ are related by $v=(R_2\theta,-R_1\theta)$, where $R_i=\partial_{x_i}(-\Delta)^{-1/2}$, the equation (1.1) is called the dissipative quasi geostrophic equation (QG). The QG equation is a model in the geophysical fluid dynamics [3]. In particular, the case $\alpha=1$ is called critical and the regularity of solutions to the QG equations is studied recently by [1, 5, 4]. If θ is a finite energy weak solution to the QG equations, it is known that $v=(R_2\theta,-R_1\theta)$ belongs to the class of bounded mean oscillation, denoted by $BMO(\mathbb{R}^d)$, for all $t>0$. This is derived from the maximum principle-type estimates for θ and the boundedness of the Riesz transformation in $BMO(\mathbb{R}^d)$. Therefore, it is important to study (1.1) under the regularity condition $v(t) \in (BMO(\mathbb{R}^d))^d$ for $t > 0$ if one takes the applications to the QG equations into account. Indeed, in [1, 4] they proved the global regularity of solutions to the critical QG equations by showing the continuity of solutions to (1.1) with $A_K(t) = (-\Delta)^{1/2}$ and with a given v in the class of $L^\infty(0,\infty;(BMO(\mathbb{R}^d))^d)$.

Motivated by the results of [1, 4] we study the fundamental solutions to (1.1) with a given v in the class of BMO. We establish pointwise upper bounds and continuity estimates of fundamental solutions associated with (1.1). Our approach is different from [1, 4], and based on the methods in [2, 6, 7] whose origin is seen in the classical results by [9] for the second-order parabolic equations of divergence forms. In particular, our results give another proof for the global regularity of the critical QG equations. The details of the arguments will be given in [8] and omitted in this note.

To describe the main results precisely, let us recall the definition of the BMO space:

$$BMO(\mathbb{R}^d) = \big\{ f \in L^1_{loc}(\mathbb{R}^d) \mid \|f\|_{BMO} = \sup_B \frac{1}{|B|} \int_B |f - \mathrm{Avg}_B f| \, \mathrm{d}x < \infty \big\}. \tag{1.5}$$

Here the supremum is taken over all balls $B = B_R(x)$ (the ball with radius $R > 0$ centered at $x \in \mathbb{R}^d$), and $|B|$ is the volume of the ball B. The value $\mathrm{Avg}_B f$ is defined by

$$\mathrm{Avg}_B f = \frac{1}{|B|} \int_B f(x) \, \mathrm{d}x. \tag{1.6}$$

In this note we impose the following two conditions on v:

$$t^{1-\frac{1}{\alpha}} v(t) \in L^\infty(0,\infty;(BMO(\mathbb{R}^d))^d), \tag{1.7}$$

$$\mathrm{div}\ v(t) = 0 \qquad \text{for a.e. } t > 0 \text{ in the sense of distributions.} \tag{1.8}$$

For simplicity of notations we will write

$$\|v\|_{X_\alpha} = \sup_{t>0} t^{1-\frac{1}{\alpha}} \|v(t)\|_{BMO}. \tag{1.9}$$

The time weight in (1.7) reflects the scaling invariant property of (1.1). Indeed, if θ is a solution to (1.1) then the rescaled function $\theta_\lambda(t,x) = \theta(\lambda t, \lambda^{1/\alpha} x)$ is also a solution to the equation of the type (1.1) with v replaced by $v_\lambda(t,x) = \lambda^{1-1/\alpha} v(\lambda t, \lambda^{1/\alpha} x)$. It is easy to see that $\|v_\lambda\|_{X_\alpha} = \|v\|_{X_\alpha}$ for all $\lambda > 0$, i.e., $\|\cdot\|_{X_\alpha}$ is a scaling invariant norm. Such a scaling invariant property is heuristically known as a key in the study of several quantities of solutions.

We first state the pointwise estimates of fundamental solutions of (1.1), denoted by $P_{K,v}(t,x;s,y)$.

Theorem 1.1 *Assume that (1.3), (1.7), and (1.8) hold. Then there exists a fundamental solution* $P_{K,v}(t,x;s,y)$ *to (1.1) such that*

$$P_{K,v}(t,x;s,y) \le C_1 (t-s)^{-\frac{d}{\alpha}}, \tag{1.10}$$

$$P_{K,v}(t,x;s,y) \le C_2 \big(1 + F[v](t,s,x,y)\big)^{d+\alpha} (t-s)^{-\frac{d}{\alpha}} \big(1 + \frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\big)^{-d-\alpha}, \tag{1.11}$$

for all $t > s \ge 0$ *and* $x,\ y \in \mathbb{R}^d$, *where*

$$F[v](t,s,x,y) = (t-s)^{-\frac{1}{\alpha}} \sup_{s<r<t} \Big| \int_s^r \mathrm{Avg}_{B_{|x-y|}(x)} v(\tau) \, \mathrm{d}\tau \Big|. \tag{1.12}$$

Here C_1 *depends only on* d, α, *and* C_0, *and* C_2 *depends only on* d, α, C_0, *and* $\|v\|_{X_\alpha}$.

Note that the constants C_1 and C_2 in Theorem 1.1 do not depend on the time variables as well as spatial variables, due to the scaling invariant assumptions on v. Est. (1.11) shows that $P_{K,v}(t,x;s,y)$ is bounded by the modification of $C(t-s)^{-\frac{d}{\alpha}}(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}})^{-d-\alpha}$, which implies that $P_{K,v}(t,x;s,y)$ has the similar pointwise decay property as the fundamental solution to the fractional heat equations

$$\partial_t\theta+(-\Delta)^{\frac{\alpha}{2}}\theta=0,\quad t>0,\quad x\in\mathbb{R}^d. \tag{1.13}$$

Indeed, the fundamental solutions to (1.13), denoted by $P_0(t,x;s,y)$, has the estimate

$$C(t-s)^{-\frac{d}{\alpha}}(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}})^{-d-\alpha}\le P_0(t,x;s,y)\le C'(t-s)^{-\frac{d}{\alpha}}(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}})^{-d-\alpha}, \tag{1.14}$$

for some positive constants C and C'.

The term $F[v]$ in (1.12) can be estimated by assuming in addition that $v(t)$ belongs to the uniformly local L^1 space: $L^1_{uloc}(\mathbb{R}^d)=\{f\in L^1_{loc}(\mathbb{R}^d)\mid \|f\|_{L^1_{uloc}}=\sup_{x\in\mathbb{R}^d}\mathrm{Avg}_{B_1(x)}|f|<\infty\}$. In addition to (1.7) and (1.8) let us assume that

$$t^{1-1/\alpha}v(t)\in L^\infty(0,\infty;(L^1_{uloc}(\mathbb{R}^d))^d). \tag{1.15}$$

Then we get the following

Corollary 1.2 *Assume that (1.3), (1.7), (1.8), and (1.15) hold. Then the fundamental solution $P_{K,v}(t,x;s,y)$ in Theorem 1.1 satisfies*

(i) *for $(t-s)^{1/\alpha}\le|x-y|\le 1$:*

$$P_{K,v}(t,x;s,y)\le C(t-s)^{-\frac{d}{\alpha}}\big(1+|\log(t-s)|\big)^{d+\alpha}\Big(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}}\Big)^{-d-\alpha}, \tag{1.16}$$

(ii) *otherwise:*

$$P_{K,v}(t,x;s,y)\le C(t-s)^{-\frac{d}{\alpha}}(1+\frac{|x-y|}{(t-s)^{\frac{1}{\alpha}}})^{-d-\alpha}, \tag{1.17}$$

Next we state the continuity results of fundamental solutions.

Theorem 1.3 *Assume that (1.3), (1.7), and (1.8) hold. Then the fundamental solution $P_{K,v}(t,x;s,y)$ in Theorem 1.1 satisfies*

$$\begin{aligned}&|P_{K,v}(t_1,x_1;s_1,y_1)-P_{K,v}(t_2,x_2;s_2,y_2)|\\ \le\;&\frac{C}{(\min\{t_1-s_1,t_2-s_2\})^{\frac{d}{\alpha}}}\left(\frac{|x_1-x_2|+|y_1-y_2|+(t_1-t_2)^{\frac{1}{\alpha}}+(s_1-s_2)^{\frac{1}{\alpha}}}{(\min\{t_1-s_1,t_2-s_2\})^{\frac{1}{\alpha}}}\right)^c.\end{aligned}$$

Here C and c depend only on d, α, C_0, and $\|v\|_{X_\alpha}$.

The proof of Theorem 1.3 is based on the arguments in [6, 7], where the convective term is not taken into account. Our approach does work for the case $\alpha\in(0,1)$, but a suitable Hölder continuity is required for v. These are also discussed in [8].

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