Blow-up Phenomena on the Curveture of the Closed Plane Elastic Curves

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Let us consider a colsed elastic plane curve Γ with length 2π. We denote its arc-length and its curveture of Γ by s and $\kappa(s)$ respectively. We define the signed area M of the domain bounded by Γ by $M := \dfrac{1}{2}\displaystyle\int_\Gamma xdy - ydx$. We consider the following problem:

For given M with $|M| < \pi$ and non-negative winding number ω, find the curve Γ such that the elastic energy $\mathcal{E}(\kappa) := \dfrac{1}{2}\displaystyle\int_0^{2\pi} \kappa^2(s)ds$ takes the minimum.

First, K. Watanabe [7]-[9] studied the case $\omega = 1$. He proved the existence of the solution and derived the Euler equation. Further he obtained the solution and showed the uniquness when Γ is close to the unit circle. W. Matsumoto, M. Murai and S. Yotsutani[1]-[3] and M. Murai [4] derived the represention fomulae of the solutions using the elliptic functions and the complete elliptic integrals, and they compeletly made clear the structure of the solutions of the Euler equation with general winding number.

The minimizer satisfies the following Euler-Lagrange equation:

$$
\text{(E)}\begin{cases}
\kappa_{ss} + \dfrac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, \quad s \in [0, 2\pi], \\
\kappa(0) = \kappa(2\pi),\ \kappa_s(0) = \kappa_s(2\pi), \\
\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \kappa(s)ds = \omega \\
\dfrac{4\pi^2\mu + \pi\displaystyle\int_0^{2\pi} \kappa(s)^2 ds}{4\pi\omega\mu + \displaystyle\int_0^{2\pi} \kappa(s)^3 ds} = M
\end{cases}
$$

where ν, μ are some constants.

We take starting point $s = 0$ as $\kappa(0) = \max_{0\le s\le 2\pi} \kappa(s)$. For any natural number n, there exist the solutions $\kappa(s)$ with the prime period $2\pi/n$. We call this solutions n-mode solutions. Furthermore $\kappa(s)$ is symmetric on [0, $2\pi/n$] at its center.

Let us investigate the limit shape of the curves. In the case of $\omega = 0$, 1-mode solution, the closed curve Γ deforms from symmetric form like 8 to non symmetric form with a large ring and a small ring as the area M increse from 0 to π. The small ring shrinks to a point and the large ring tends to the unit circle. That is, the absolute value of the curvature of the small ring increases to infinity and that of large ring tends to 1. The same phenomenon arises when the area M decrease to $-\pi$. This kind of the blow up phenomena also occur in case of general winding number.

On this study, in the case of $\omega = 0$ and of the 1-mode solution, we prove mathematicaly the blow up phenomena, and obtain the order of the blow up. The key of the proof dues to representation fomulae for all solutions of the Euler equatoin derieved by Matsumoto-Murai-Yotsutani and to refinement of asymptotic behavior of the complete integral of the 3rd kind obtained by T. Wakasa [5], and T. Wakasa-S. Yotsutani [6].

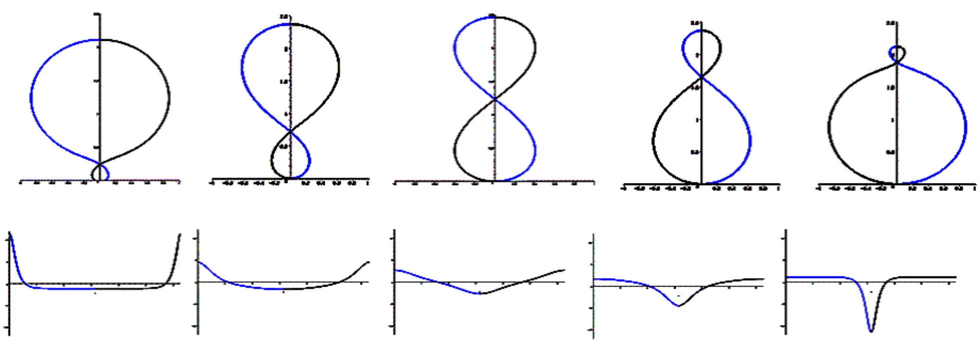
Theorem. Let $\omega = 0$. The 1-mode solution of (E) is unique.We write this solution $\kappa(s; M)(|M| < \pi)$. Then,following holds.

$$\lim_{M\uparrow\pi} \kappa(s; M) = \begin{cases} 1 & s \in [0, \pi) \cup (\pi, 2\pi] \quad \text{uniformly in the wider sence,} \\ -\infty & s = \pi. \end{cases}$$

$$\lim_{M\downarrow-\pi} \kappa(s; M) = \begin{cases} \infty & s = 0, \\ -1 & s \in (0, 2\pi) \quad \text{uniformly in the wider sence.} \end{cases}$$

Futhermore,

$$\kappa(\pi, M) = \frac{8}{M - \pi} + O(1) \quad (\text{as} \quad M \uparrow \pi),$$
$$\kappa(0; M) = \frac{8}{M + \pi} + O(1) \quad (\text{as} \quad M \downarrow -\pi).$$



M=-2.4 M=-1.21 M=0 M=1.21 M=2.4

Figure 1: closed curve Γ and its curvature $\kappa(s)$

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