UNIQUENESS OF SOLUTIONS FOR NON-CUTOFF BOLTZMANN EQUATION AND SINGULAR CHANGE OF VARIABLES IN PRES-POST COLLISIONAL VELOCITIES

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ABSTRACT. In this note we consider the uniqueness of solution to the Cauchy problem for non-cutoff Boltzmann equation in the whole space. Several results in different function spaces are detailed in the cases of hard and soft potentials. In particular, we discuss the uniqueness of the solution with the polynomial decay with respect to the velocity variable, in the soft potential case of the classical sense, where the singular change of variables from "pres" to "post " collisional velocity plays an important role.

1. INTRODUCTION AND UNIQUENESS RESULTS

We consider the Cauchy problem for the spatially inhomogeneous Boltzmann equation,

$$\left\{ \begin{array}{l} \partial_t f + v\cdot\nabla_x f = Q(f,f), \;\; x, v\in\mathbb{R}^3, \;\; t>0, \\ f(0,x,v) = f_0(x,v), \end{array}\right. \tag{1}$$

where $f=f(t,v)$ is the density distribution function of particles with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$ at time t. The right hand side of (1) is given by the Boltzmann bilinear collision operator

$$Q(g,f)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{g(v'_*)f(v')-g(v_*)f(v)\right\}d\sigma dv_*,$$

which is well-defined for suitable functions f and g specified later. Notice that the collision operator $Q(\cdot,\cdot)$ acts only on the velocity variable $v\in\mathbb{R}^3$. In the following discussion, we will use the $\sigma-$representation, that is, for $\sigma\in\mathbb{S}^2$,

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\;\; v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma,$$

which give the relations between the post and pres collisional velocities. The non-negative cross section $B(z,\sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|}\cdot\sigma$. In what follows we assume that it takes the form

$$B(|v-v_*|,\cos\theta)=\Phi(|v-v_*|)b(\cos\theta),\;\; \cos\theta=\frac{v-v_*}{|v-v_*|}\cdot\sigma,\; 0\le\theta\le\frac{\pi}{2},$$

where the angular factor $b(\cos\theta)$ is assumed to have the following singularity;

$$\sin\theta\, b(\cos\theta)\;\approx\; K\theta^{-1-2s},\;\text{when}\;\theta\to 0+, \tag{2}$$

for $0<s<1$ and a constant $K>0$, and the kinetic factor $\Phi=\Phi_\gamma$ is given by

$$\Phi_\gamma(|v-v_*|)=|v-v_*|^\gamma, \tag{3}$$

for some $\gamma > \max\{-3, -3/2-2s\}$. If the inter-molecule potential satisfies the inverse power law potential $U(\rho) = \rho^{-(q-1)}, q > 2$ (, where ρ denotes the distance between two interacting molecules), then

$$\Phi(|v-v_*|) = |v-v_*|^{(q-5)/(q-1)} \text{ and}$$
$$\sin\theta\, b(\cos\theta) \approx K\theta^{-1-2s} \text{ as } \theta \to 0,$$

where $K > 0$ and $0 < s = 1/(q-1) < 1$. Namely, for this physical case, we have

$$\gamma = \frac{q-5}{q-1} = 1 - 4\frac{1}{q-1} = 1-4s$$

which is contained in our assumptions $0 < s < 1$ and $\gamma > \max\{-3, -3/2-2s\}$.

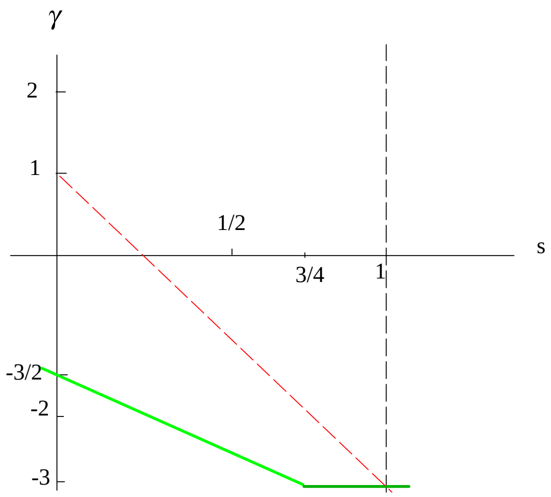

FIGURE 1. (s,γ)

We use the usual function spaces as follows: For $p \geq 1$ and $\beta \in \mathbb{R}$, we set

$$\|f\|_{L^p_\beta} = \left(\int_{\mathbb{R}^3} |\langle v\rangle^\beta f(v)|^p dv\right)^{1/p},$$

and for $s \in \mathbb{R}$

$$\|f\|_{H^s_\beta(\mathbb{R}^3_v)} = \Big(\int_{\mathbb{R}^3} |\langle D_v\rangle^s \big(\langle v\rangle^\beta f(v)\big)|^2 dv\Big)^{1/2}.$$

Furthermore

$$\|f\|_{H^s_\beta(\mathbb{R}^6_{x,v})} = \Big(\int_{\mathbb{R}^6} |\langle D_x, D_v\rangle^s \big(\langle v\rangle^\beta f(x,v)\big)|^2 dv dx\Big)^{1/2}.$$

For the uniqueness of solution, we first consider the function space with polynomial decay in the velocity variable. For $m \in \mathbb{R}$ and $\ell \geq 0$, set

$$\tilde{\mathscr{P}}^{m,\ell}_0(\mathbb{R}^6_{x,v}) = \Big\{g \in \mathscr{D}'(\mathbb{R}^6_{x,v});\, g \in L^\infty(\mathbb{R}^3_x; H^m_\ell(\mathbb{R}^3_v))\Big\},$$

and for $T > 0$

$$\tilde{\mathscr{P}}^{m,\ell}([0,T]\times\mathbb{R}^6_{x,v}) = \Big\{f \in C^0([0,T]; \mathscr{D}'(\mathbb{R}^6_{x,v})); \quad s.t.\ f \in L^\infty([0,T]\times\mathbb{R}^3_x;\, H^m_\ell(\mathbb{R}^3_v))\Big\}.$$

Our first theorem concerns the uniqueness of solution for the case $\gamma \le 0$, which is called *soft potential case* in the classical sense and *Maxwellian molecule type.*

Theorem 1. *Assume that* $0 < s < 1$ *and*

$$\max\big(-3, -3/2-2s\big)<\gamma\le 0\,.$$

Let $0 < T < +\infty$ *and let* $\ell_0 \ge 7$. *Suppose that the Cauchy problem* (1) *admits two solutions* $f_1(t), f_2(t) \in \tilde{\mathscr{P}}^{2s,\ell_0}([0,T]\times\mathbb{R}^6_{x,v})$ *for the same initial datum* $f_0 \in \tilde{\mathscr{P}}^{0,0}_0(\mathbb{R}^6_{x,v})$. *If one solution is non-negative then* $f_1(t) \equiv f_2(t)$.

For the uniqueness of solution in the case $\gamma > 0$, we consider the function space with exponential decay in the velocity variable. More precisely, for $m \in \mathbb{R}$, set

$$\tilde{\mathscr{E}}^m_0(\mathbb{R}^6) = \Big\{g \in \mathscr{D}'(\mathbb{R}^6_{x,v});\, \exists \rho_0 > 0\ s.t.\ e^{\rho_0\langle v\rangle^2} g \in L^\infty(\mathbb{R}^3_x; H^m(\mathbb{R}^3_v))\Big\},$$

and for $T > 0$

$$\begin{aligned}\tilde{\mathscr{E}}^m([0,T]\times\mathbb{R}^6_{x,v}) \;=\; &\Big\{f \in C^0([0,T];\mathscr{D}'(\mathbb{R}^6_{x,v}));\, \exists \rho > 0\\ &\quad s.t.\ e^{\rho\langle v\rangle^2} f \in L^\infty([0,T]\times\mathbb{R}^3_x;\, H^m(\mathbb{R}^3_v))\Big\}.\end{aligned}$$

Theorem 2. *Assume that* $0 < s < 1$ *and*

$$\max\big(-3, -3/2-2s\big)<\gamma< 2-2s\,.$$

Let $0 < T < +\infty$ *and suppose that the Cauchy problem* (1) *admits two solutions* $f_1(t), f_2(t) \in \tilde{\mathscr{E}}^{2s}([0,T]\times\mathbb{R}^6_{x,v})$ *for the same initial datum* $f_0 \in \tilde{\mathscr{E}}^0_0(\mathbb{R}^6)$. *If one solution is non-negative then* $f_1(t) \equiv f_2(t)$.

In the above theorem we can relax the assumptions of the regularity index $2s$ and the non-negativity on solutions.

Theorem 3. *Assume that* $0 < s < 1$ *and*

$$\max\big(-3, -3/2-2s\big)<\gamma< 2-2s\,.$$

Suppose that the Cauchy problem (1) *admits two solutions* $f_1(t), f_2(t) \in \tilde{\mathscr{E}}^{s}([0,T]\times\mathbb{R}^6_{x,v})$ *for the same initial datum* $f_0 \in \tilde{\mathscr{E}}^0_0(\mathbb{R}^6)$.
(I) *If* $f_1(t) \ge 0$ *and if there exist* $c_0, C > 0$ *independent of* $t \in]0,T[$ *such that the coercive estimate*

$$-(Q(f_1(t),h),h)_{L^2(\mathbb{R}^6)} \ge c_0\|h\|^2_{L^2(\mathbb{R}^3_x;H^s_{\gamma/2})} - C\|h\|^2_{L^2(\mathbb{R}^3_x;L^2_{(s+\gamma/2)^+}(\mathbb{R}^3_v))} \tag{4}$$

holds for any $h \in \mathscr{S}^\infty(\mathbb{R}^6)$, *then* $f_1(t) \equiv f_2(t)$.
(II) *The same conclusion as in* (I) *holds without the non-negativity of* $f_1(t)$ *if* $f_1(t)$ *satisfies the following strong coercive estimate*

$$-(Q(f_1(t),h),h)_{L^2(\mathbb{R}^6)} \ge c_0\int |||h|||^2_{\Phi_\gamma}dx - C\|h\|^2_{L^2(\mathbb{R}^3_x;L^2_{(s+\gamma/2)^+}(\mathbb{R}^3_v))}\,. \tag{5}$$

As for the terminology of the *strong* coercive estimate in the above theorem we notice that

$$\|g\|^2_{L^2_{s+\gamma/2}} + \|g\|^2_{H^s_{\gamma/2}} \lesssim |||g|||^2_{\Phi_\gamma} \lesssim \|g\|^2_{H^s_{s+\gamma/2}}\,.$$

Here

$$|||g|||^2_{\Phi_\gamma} = \int b(\cos\theta)\Phi_\gamma(|v-v_*|)\mu_*(g'-g)^2 dvdv_*d\sigma$$
$$+ \int b\Phi_\gamma(|v-v_*|)g_*^2(\sqrt{\mu}'-\sqrt{\mu})^2 dvdv_*d\sigma$$
$$= J_1(g)+J_2(g)\,,$$

$$\|g\|^2_{H^s_{\gamma/2}} \lesssim J_1(g)+\|g\|^2_{L^2_{s+\gamma/2}} \lesssim \|g\|^2_{H^s_{s+\gamma/2}}\,,\;\; J_2(g)\sim \|g\|_{L^2_{s+\gamma/2}}.$$

The case $\gamma+2s\geq 2$ is out of the physical case coming from the inverse power law potential (see the figure below).

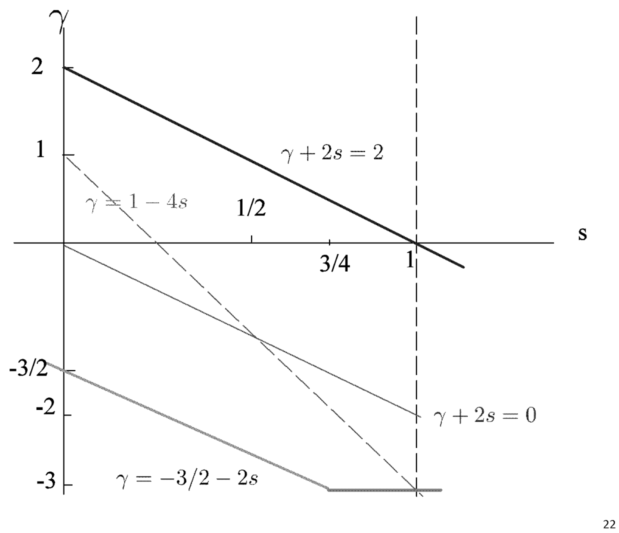

FIGURE 2. (s,γ)

However, the uniqueness of solutions holds also in the case $\gamma+2s\geq 2$ if we confine ourselves to the solution which is the perturbation around a normalized Maxwellian distribution $\mu(v)=e^{-|v|^2/2}/(2\pi)^{3/2}$, that is,

$$f=\mu+\mu^{1/2}\tilde g\,.$$

Theorem 4. *Assume that* $0<s<1$ *and* $\gamma+2s\geq 2$. *Let* $\ell_1>3/2+\gamma+2s$. *Then there exists an* $\varepsilon_0>0$ *satisfying the following: Let* $f_1(t), f_2(t)\in\tilde{\mathscr{E}}^s([0,T]\times\mathbb{R}^6_{x,v})$ *be two solutions of the Cauchy problem* (1) *and satisfy*

$$\mu^{-1/2}\big(f_j(t)-\mu\big)\in L^\infty([0,T]\times\mathbb{R}^3_x;H^s_{\ell_1})\,,\;\; j=1,2\,.$$

If $\|\mu^{-1/2}\big(f_1(t)-\mu\big)\|_{L^\infty([0,T]\times\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}<\varepsilon_0$ *then* $f_1(t)\equiv f_2(t)$ *for all* $t\in[0,T]$.

In the case $\gamma+2s\le 0$ we can refine the second part (II) of Theorem 3, that is, we can consider the uniqueness in another function space

$$\tilde{\mathscr{B}}^s([0,T]\times\mathbb{R}^6_{x,v})=\Big\{f\in C^0([0,T];\mathscr{D}'(\mathbb{R}^6_{x,v}));\,\exists\rho>0$$
$$s.t.\ e^{\rho\langle v\rangle^2}f\in L^\infty([0,T]\times\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))\cap L^2([0,T];L^\infty(\mathbb{R}^3_x;H^s(\mathbb{R}^3_v)))\Big\},$$

which is wider than

$$\tilde{\mathscr{E}}^s([0,T]\times\mathbb{R}^6_{x,v}) = \Big\{f\in C^0([0,T];\mathscr{D}'(\mathbb{R}^6_{x,v}));\,\exists\rho>0$$
$$s.t.\ e^{\rho\langle v\rangle^2}f\in L^\infty([0,T]\times\mathbb{R}^3_x;\,H^s(\mathbb{R}^3_v))\Big\}.$$

Theorem 5. *Assume that* $0<s<1$ *and*

$$\max\big(-3,-3/2-2s\big)<\gamma\le -2s\,.$$

Let $0<T<+\infty$ *and suppose that* $f_1(t)\in\tilde{\mathscr{B}}^s([0,T]\times\mathbb{R}^6_{x,v})$ *is a solution to the Cauchy problem* (1) *satisfying the strong coercive estimate* (5)*. Then* $f_1(t)$ *coincides with any another solution* $f_2(t)\in\tilde{\mathscr{B}}^s([0,T]\times\mathbb{R}^6_{x,v})$*.*

2. UNIQUENESS OF KNOWN SOLUTIONS

Theorems announced in the preceding section are applicable to show the uniqueness of known solutions given in [5, 6, 8] in the wider spaces than those where they are constructed.

Uniqueness of global solutions for small initial data

• If $\gamma+2s>0$, $0<s<1$ and if $\|\tilde g_0\|_{H^k_{\ell_1}(\mathbb{R}^6)}$ ($k\ge 6,\ell_1>3/2+2s+\gamma$) is small enough then (see [6]) the Cauchy problem (1) with the initial datum $\mu+\sqrt{\mu}\tilde g_0$ admits a global solution $f_1(t)$ of the form $\mu+\sqrt{\mu}\tilde g(t,x,v)$ with $\tilde g\in L^\infty([0,\infty[;H^k_{\ell_1}(\mathbb{R}^6))$ and its norm $\|\tilde g\|_{L^\infty([0,\infty[;H^k_{\ell_1}(\mathbb{R}^6))}$ small. Furthermore this $f_1(t)$ satisfies the strong coercive estimate (5).

The part (II) of Theorem 3 shows the uniqueness of $f_1(t)$ in the function space $\tilde{\mathscr{E}}^s([0,T]\times\mathbb{R}^6_{x,v})$ for any $T>0$, provided that $\gamma+2s<2$.

$$\tilde{\mathscr{E}}^s([0,T]\times\mathbb{R}^6_{x,v}) = \Big\{f\in C^0([0,T];\mathscr{D}'(\mathbb{R}^6_{x,v}));\,\exists\rho>0$$
$$s.t.\ e^{\rho\langle v\rangle^2}f\in L^\infty([0,T]\times\mathbb{R}^3_x;\,H^s(\mathbb{R}^3_v))\Big\}.$$

When $\gamma+2s\ge 2$, Theorem 4 shows $f_1(t)$ coincides with another solution of the form $\mu+\sqrt{\mu}\tilde g(t,x,v)$ with $\tilde g\in L^\infty([0,\infty[;H^s_{\ell_1}(\mathbb{R}^6))$.

• If $\max(-3,-3/2-2s)<\gamma\le-2s$ and $0<s<1$ then (see [5]) a global solution is given of the form $\mu+\sqrt{\mu}\tilde g$ with

$$\tilde g(t,x,v)\in L^\infty([0,\infty[;\tilde{\mathscr{H}}^N_\ell(\mathbb{R}^6_{x,v}))\,,\ \ N\ge 6\,,\ \ \ell\ge N$$

where

$$\tilde{\mathscr{H}}^N_\ell(\mathbb{R}^6_{x,v}))=\{f\in\mathscr{S}'(\mathbb{R}^6_{x,v});\,\|f\|_{\tilde{\mathscr{H}}^N_\ell(\mathbb{R}^6_{x,v})}$$
$$=\sum_{|\alpha+\beta|\le N}\|\langle v\rangle^{|s+\gamma/2|(\ell-|\beta|)}\partial^\alpha_x\partial^\beta_v f\|_{L^2(\mathbb{R}^6)}<\infty\}\,,$$

provided that the initial datum $f_0(x,v)=\mu+\sqrt{\mu}\tilde g_0(x,v)$ and $\|\tilde g_0\|_{\tilde{\mathscr{H}}^N_\ell(\mathbb{R}^6_{x,v})}$ is small enough. Since $\|\tilde g(t)\|_{\tilde{\mathscr{H}}^N_\ell(\mathbb{R}^6_{x,v})}$ is small, the solution $f_1(t)=\mu+\sqrt{\mu}\tilde g(t,x,v)$ satisfies also the strong coercive estimate (5) and hence (II) of Theorem 3 shows the uniqueness of the solution in

$\tilde{\mathscr{E}}^s([0,T]\times\mathbb{R}^6_{x,v})$. If the initial datum $f_0\ge 0$ then Theorem 1 yields the uniqueness in the space $\tilde{\mathscr{P}}^{2s,\ell_0}([0,T]\times\mathbb{R}^6_{x,v})$.

• When $\max(-3,-3/2-2s)<\gamma\le -2s$ and $0<s<1$, we have *another type* global solution of the form $\mu+\sqrt{\mu}\tilde{g}$ with

$$\tilde{g}(t,x,v)\in L^\infty([0,\infty[;H^N(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))) \\ \cap L^2_{\ell oc}([0,\infty[;H^N(\mathbb{R}^3_x;H^s(\mathbb{R}^3_v)))\,,\quad N\ge 3$$

if the initial datum $f_0(x,v)=\mu+\sqrt{\mu}\tilde{g}_0(x,v)$ and $\|\tilde{g}_0\|_{H^N(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}$ is small enough (see [5]). By the Sobolev embedding, the solution $f_1(t)=\mu+\sqrt{\mu}\tilde{g}(t)$ belongs to $\tilde{\mathscr{B}}^s([0,T]\times\mathbb{R}^6_{x,v})$ for any $T>0$. Since the smallness of $\|\tilde{g}\|_{L^\infty([0,\infty[\times\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}$ implies the strong coercive estimate (5), the solution $f_1(t)$ is unique in $\tilde{\mathscr{B}}^s([0,T]\times\mathbb{R}^6_{x,v})$ for any $T>0$, by means of Theorem 5.

Uniqueness of local solutions for non-small initial data

• Suppose that $-3/2<\gamma<1-2s$ and $0<s<1/2$. In [8], bounded solutions of the Boltzmann equation in the whole space have been constructed without specifying any limit behaviors at the spatial infinity and without assuming the smallness condition on initial data. More precisely, it has been shown that if the initial datum is non-negative and belongs to a uniformly local Sobolev space

$$H^k_{ul}(\mathbb{R}^6)=\{g\mid \|g\|^2_{H^k_{ul}(\mathbb{R}^6)} \\ =\sum_{|\alpha+\beta|\le k}\sup_{a\in\mathbb{R}^3}\int_{\mathbb{R}^6}|\phi_1(x-a)\partial_x^\alpha\partial_v^\beta g(x,v)|^2dxdv<+\infty\}.$$

with the Maxwellian decay property in the velocity variable, then the Cauchy problem of the Boltzmann equation possesses a non-negative local solution in the same function space. Since solutions there are non-negative and belong to $\mathscr{P}^{2s,\ell_0}([0,T]\times\mathbb{R}^6_{x,v})$, Theorem 1 shows their uniqueness when $\gamma\le 0$. For the case $\gamma>0$,Theorem 2 is applicable.

3. SINGULAR CHANGE OF VARIABLES AND PROOF OF THEOREM 1

In this section we explain the key point of the proof of Theorem 1. Differing from Theorems 2-5 where some power of the Maxwellian weight, that is, $e^{-\rho|v|^2},\rho>0$, absorbs the moments $\langle v\rangle$ in getting the favorable estimates (see [7]), we need more precise estimations in the proof of Theorem 1, by using the singular change of the variables in pres-post collisional velocity introduced in [10] as follows:

$$v_*\mapsto v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma\,,$$

where the Jacobian is computed as

$$\Big|\frac{\partial v_*}{\partial v'}\Big|=\frac{8}{\Big|I-\mathbf{k}\otimes\sigma\Big|}=\frac{8}{|1-\mathbf{k}\cdot\sigma|}=\frac{4}{\sin^2(\theta/2)}\ge 16\theta^{-2},\quad \theta\in[0,\pi/2].$$

After this change of variables, $\mathbf{k}=(v-v_*)/|v-v_*|$ is a function of v,v',σ, so that θ plays no longer the role of polar angle. In fact, "pole $\mathbf{k}$" moves with σ and hence the measure $d\sigma$ is no longer given by $\sin\theta d\theta d\phi$.Hence we need a new pole, independent of σ. A possible choice is now $\mathbf{k}''=(v'-v)/|v'-v|$, for which the polar angle ψ defined by $\cos\psi=k''\cdot\sigma$ satisfies,

$$\psi=\frac{\pi}{2}-\frac{\theta}{2},\quad d\sigma=\sin\psi d\psi d\phi,\quad \psi\in[\frac{\pi}{4},\frac{\pi}{2}].$$

The total singularity arising from $b(\cos\theta)d\sigma$ becomes $\theta^{-2-2-2s}$, which is bigger than (2) (see Figure 3 below).

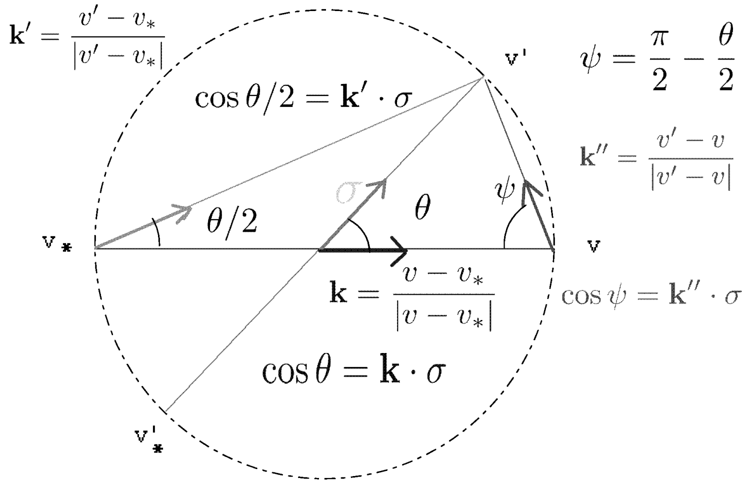

FIGURE 3. pres-post collisional velocities

For the proof of the uniqueness, we set $\varphi(v,x) = (1+|v|^2+|x|^2)^{\alpha/2}$ and

$$W_{\varphi,\ell} = \frac{\langle v\rangle^\ell}{\varphi(v,x)} = \frac{(1+|v|^2)^{\ell/2}}{(1+|v|^2+|x|^2)^{\alpha/2}}\,, \quad \alpha > 3/2.$$

Set $F = f_1 - f_2$. Then it follows from (1) that

$$\begin{cases} F_t + v\cdot\nabla_x F = Q(f_1, F) + Q(F,\, f_2)\,, \\ F|_{t=0} = 0. \end{cases} \tag{6}$$

Let $S(\tau) \in C_0^\infty(\mathbb{R})$ satisfy $0 \le S \le 1$ and

$$S(\tau) = 1,\ \ |\tau| \le 1\ \ ;\ \ S(\tau) = 0,\ \ |\tau| \ge 2.$$

Set $S_N(D_x) = S(2^{-2N}|D_x|^2)$ and multiply $W_{\varphi,l}S_N(D_x)^2W_{\varphi,l}F$ by (6). Integrating and letting $N \to \infty$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6)} &= \Big(W_{\varphi,l}\,Q(f_1,F) + W_{\varphi,l}\,Q(F,\,f_2)\,, W_{\varphi,l}F\Big)_{L^2(\mathbb{R}^6)} \\ &\quad - (v\cdot\nabla_x(\varphi^{-1})W_lF, W_{\varphi,l}F)_{L^2(\mathbb{R}^6)},\end{aligned}$$

because $(v\cdot\nabla_x S_N(D_x)W_{\varphi,l}F, S_N(D_x)W_{\varphi,l}F)_{L^2(\mathbb{R}^6)} = 0$. The second term on the right hand side is estimated by $\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)}$ because

$$|v\cdot\nabla_x(\varphi^{-1})| \lesssim \varphi^{-1}.$$

If $f_1 \geq 0$ then we have

$$\left(W_{\varphi,l}Q(f_1,F)\,,W_{\varphi,l}F\right)_{L^2(\mathbb{R}^6)} \lesssim \|f_1\|_{L^\infty(\mathbb{R}^3_x,H^{2s}_{\ell+3/2+\varepsilon}(\mathbb{R}^3_v))}\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)}\,. \tag{7}$$

On the other hand,

$$\left(W_{\varphi,l}Q(F,f_2)\,,W_{\varphi,l}F\right)_{L^2(\mathbb{R}^6)} \lesssim \|f_2\|_{L^\infty(\mathbb{R}^3_x,H^{2s}_{\ell+2s}(\mathbb{R}^3_v))}\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6)}\,. \tag{8}$$

Once we would admit those estimates we could obtain

$$\begin{aligned}&\frac{d}{dt}\|W_{\varphi,l}F(t)\|^2_{L^2(\mathbb{R}^6)}\\ &\lesssim \left(\|f_1\|_{L^\infty([0,T]\times\mathbb{R}^3_x,H^{2s}_{\ell+3/2+\varepsilon}(\mathbb{R}^3_v))}+\|f_2\|_{L^\infty([0,T]\times\mathbb{R}^3_x,H^{2s}_{\ell+2s}(\mathbb{R}^3_v))}\right)\|W_{\varphi,l}F\|^2_{L^2(\mathbb{R}^6)}\,,\end{aligned}$$

which could concludes the proof of Theorem 1.

The first estimate (7) is a consequence of the following two lemmas.

Lemma 6. *Let $0<s<1$ and let $0\geq\gamma>\max\{-3,-2s-3/2\}$. Then we have*

$$\left(Q(f,h),h\right)_{L^2(\mathbb{R}^6)} \leq -\frac{1}{2}\int \mathscr{D}(f,h)dx + C\|f\|_{L^\infty(\mathbb{R}^3_x,H^{2s'}_{3/2+\varepsilon}(\mathbb{R}^3_v))}\|h\|^2_{L^2(\mathbb{R}^6)}\,,$$

where $s'\geq 0$ satisfies $\gamma+2s'>-3/2$ and $s'<\min\{s,3/4\}$. Here

$$\mathscr{D}(f,h) = \iiint Bf(h-h')^2 dv dv_* d\sigma\,.$$

Lemma 7. *Let $\ell\geq 6$. Then*

$$\begin{aligned}&\left|\Big(\big(W_{\varphi,l}\,Q(f,\,g)-Q(f,\,W_{\varphi,l}\,g)\big),\;h\Big)_{L^2(\mathbb{R}^6)}\right|\\ &\lesssim \|f\|_{L^\infty(\mathbb{R}^3_x;H^{(2s-1)^+}_{3/2+\varepsilon}(\mathbb{R}^3_v))}\|h\|^2_{L^2(\mathbb{R}^6)}\\ &\quad+\left(\int\mathscr{D}(|f|,h)dx\right)^{1/2}\|f\|^{1/2}_{L^\infty(\mathbb{R}^3_x;L^2_{\ell+3/2+\varepsilon}(\mathbb{R}^3_v))}\|W_{\varphi,\ell}\,g\|_{L^2(\mathbb{R}^6)}.\end{aligned} \tag{9}$$

The second estimate (8) is reduced to

$$\left(W_{\varphi,l}Q(f,g)\,,h\right)_{L^2(\mathbb{R}^3_v)} \lesssim \|g\|_{H^{2s}_{\ell+2s}(\mathbb{R}^3_v))}\|W_{\varphi,l}\,f\|_{L^2(\mathbb{R}^3_v)}\|h\|_{L^2(\mathbb{R}^3_v)}$$

if we regard x as a parameter. To prove this estimate, let $0\leq\phi(z)\leq 1$ be a smooth radial function with value 1 for z close to 0, and 0 for large values of z. Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z)+\Phi_\gamma(z)(1-\phi(z)) = \Phi_{sing}(z)+\Phi_{reg}(z).$$

Then we write

$$Q(f,g) = Q_{sing}(f,g)+Q_{reg}(f,g),$$

where the kinetic factor in the collision operator is defined according to the decomposition respectively. In what follows we consider

$$\left(W_{\varphi,l}Q_{reg}(f,g)\,,h\right)_{L^2(\mathbb{R}^3_v)} \lesssim \|g\|_{H^{2s}_{\ell+2s}(\mathbb{R}^3_v))}\|W_{\varphi,l}\,f\|_{L^2(\mathbb{R}^3)}\|h\|_{L^2(\mathbb{R}^3_v)}\,,$$

though the singular part Q_{sing} also requires fairly long computations (see [7]). Write

$$\begin{aligned}\left(W_{\varphi,l}\,Q_{reg}(f,\,g)\,,h\right)_{L^2(\mathbb{R}^3)} &= \left(\,Q_{reg}(f,\,W_{\varphi,l}\,g)\,,h\right)_{L^2(\mathbb{R}^3)}\\ &\quad+\Big(\big(W_{\varphi,l}\,Q_{reg}(f,\,g)-Q_{reg}(f,\,W_{\varphi,l}\,g)\big),\;h\Big)_{L^2(\mathbb{R}^3)} = A+B\,.\end{aligned}$$

By the upper bound estimate in Theorem 2.1 of [3], we have

$$\begin{aligned}\int |A| dx &\lesssim \int \|f\|_{L^1_{(\gamma+2s)^+}(\mathbb{R}^3_v)} \|W_{\varphi,l} g\|_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v)} \|h\|_{L^2(\mathbb{R}^3_v)} dx \\ &\lesssim \int \|\frac{\varphi}{\langle x\rangle^\alpha}\frac{f}{\varphi}\|_{L^2_{2s+3/2+\varepsilon}(\mathbb{R}^3_v)} \|\frac{\langle x\rangle^\alpha}{\varphi} W_l g\|_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v)} \|h\|_{L^2(\mathbb{R}^3_v)} dx \\ &\lesssim \|W_{\varphi,\alpha} f\|_{L^2(\mathbb{R}^6)} \|g\|_{L^\infty(\mathbb{R}^3_x; H^{2s}_{l+2s}(\mathbb{R}^3_v))} \|h\|_{L^2(\mathbb{R}^6)}.\end{aligned}$$

Here we have used $\frac{\varphi}{\langle x\rangle^\alpha} \lesssim \langle v\rangle^\alpha$. The proof of the second estimate (8) is complete because we have the following lemma as for B.

Lemma 8. *Let* $\ell \geq 5$. *Then*

$$\begin{aligned}&\left|\Big(W_{\varphi,l}\, Q_{reg}(f,\, g) - Q_{reg}(f,\, W_{\varphi,l}\, g),\ h\Big)_{L^2(\mathbb{R}^3_v)}\right| \\ &\lesssim \|W_{\varphi,l} f\|_{L^2(\mathbb{R}^3_v)} \|g\|_{H^s_{l+s}(\mathbb{R}^3_v)} \|h\|_{L^2(\mathbb{R}^3_v)}.\end{aligned}$$

Proof. Note that

$$\begin{aligned}&\Big(W_{\varphi,l}\, Q_{reg}(f,\, g) - Q_{reg}(f,\, W_{\varphi,l}\, g),\ h\Big)_{L^2(\mathbb{R}^3_v)} \\ &= \iiint b\Phi_{reg}\Big(\big(W_{\varphi,l}\big)' - \big(W_{\varphi,l}\big)\Big) f_* g' h' dv dv_* d\sigma\,,\end{aligned}$$

$$\begin{aligned}&\left|W_{\varphi,l} - W'_{\varphi,l} - \Big(\nabla_v W_{\varphi,l}\Big)(v')\cdot(v-v')\right| \\ &= \left|\int_0^1 \tau \nabla^2 W_{\varphi,l}(v+\tau(v'-v)) d\tau (v-v')^2\right| \\ &\lesssim \sin^2\left(\frac{\theta}{2}\right)\frac{W_l + W_{l-2-\alpha} W_{2+\alpha,*}}{\varphi(v_*,x)} + \sin^{l-\alpha}\left(\frac{\theta}{2}\right)\frac{W_{l,*}}{\varphi(v_*,x)} \\ &\lesssim \theta^2 W_l W_{\varphi,2+\alpha,*} + \theta^{l-\alpha} W_{\varphi,l,*},\end{aligned}$$

where $W_{\varphi,l,*} = \dfrac{W_{l,*}}{\varphi(v_*,x)}$. We consider the second order term in the Taylor expansion

$$\begin{aligned}&\iiint b\,\theta^2 \big|\big(W_{\varphi,2+\alpha} f\big)_*\big|\big|\big(W_\ell g\big)\big||h'| dv dv_* d\sigma \\ &+ \iiint b\,\theta^{\ell-2}\big|\big(W_{\varphi,\ell} f\big)_*\big||gh'| dv dv_* d\sigma \\ &= M_1 + M_2\,.\end{aligned}$$

By the Cauchy-Schwarz inequality we get

$$\begin{aligned}|M_1|^2 &\lesssim \iiint b\,\theta^2 \big|\big(W_{\varphi,2+\alpha} f\big)_*\big|\big(W_\ell g\big)^2 dv dv_* d\sigma \\ &\times \iiint b\,\theta^2 \big|\big(W_{\varphi,2+\alpha} f\big)_*\big|(h')^2 dv dv_* d\sigma \\ &\lesssim \|W_{\varphi,2+\alpha} f\|^2_{L^1(\mathbb{R}^3_v)} \|g\|^2_{L^2_\ell(\mathbb{R}^3_v)} \|h\|^2_{L^2(\mathbb{R}^3_v)}\,.\end{aligned}$$

Here we used the regular change of variable

$$v \to v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\,,$$

whose Jacobian

$$\Big|\frac{\partial v}{\partial v'}\Big| = \frac{8}{\Big|I + \mathbf{k}\otimes\sigma\Big|} = \frac{8}{|1+\mathbf{k}\cdot\sigma|} = 4/\cos^2(\theta/2) \le 8\,.$$

By the Cauchy-Schwarz inequality again we have

$$|M_2|^2 \lesssim \Big(\iiint b\,\theta^{\ell-\alpha-3/2}|g|\big(W_{\varphi,\ell} f\big)^2_* dv_* dv d\sigma\Big)$$
$$\times\Big(\iiint b\,\theta^{\ell-\alpha+3/2}|g||h'|^2 dv_* dv d\sigma\Big)$$
$$\lesssim \|g\|^2_{L^1(\mathbb{R}^3_v)}\|W_{\varphi,\ell} f\|^2_{L^2(\mathbb{R}^3_v)}\|h\|^2_{L^2(\mathbb{R}^3_v)}\,.$$

if we choose ℓ so that

$$\ell-\alpha-3/2-1-2s = \ell-\alpha+3/2-(2+2s+2) > -1\,.$$

Here we have used singular change of variables $v_* \to v'$ whose Jacobian

$$\Big|\frac{\partial v_*}{\partial v'}\Big| = \frac{4}{\sin^2(\theta/2)}$$

and the fact that the total angular singularity of the second factor is $\theta^{-(2+2s+2)}$. The first order term of the Taylor expansion can be estimated by using the symmetry property on the $\mathbb{S}^2$. The detail is omitted (see [9]). □

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