Blow up Phenomena
on the curvature of closed plane elastic curves
with the winding number $\omega \geq 1$

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Let Ω be a domain in $\mathbf{R}^2$ with smooth boundary Γ. The length of Γ is equal to 2π and the winding number $\omega \geq 0$. Let s be the arc-length parameter of Γ, $\kappa(s)$ the curvature of Γ and M the area of Ω defined by

$$M := \frac{1}{2}\int_\Gamma xdy - ydx \qquad (x, y) \in \Gamma.$$

We consider the variational problem (VP): For given $\omega \geq 0$, $L > 0$ and M with $-\pi < M < \pi$ and $\omega M \neq \pi$, find the curvature $\kappa(s)$ such that the functional $\mathcal{E}(\kappa) := \frac{1}{2}\int_0^{2\pi} \kappa(s)^2 ds$ takes the minimum.

The minimizer satisfies the following Euler-Lagrange equation:

$$(E)\begin{cases} \kappa_{ss} + \frac{1}{2}\kappa^3 + \mu\kappa - \nu = 0, \quad s \in [0, 2\pi], \\ \kappa(0) = \kappa(2\pi),\ \kappa_s(0) = \kappa_s(2\pi), \\ \frac{1}{2\pi}\int_0^{2\pi} \kappa(s)ds = \omega \\ \dfrac{4\pi^2\mu + \pi\displaystyle\int_0^{2\pi} \kappa(s)^2 ds}{4\pi\omega\mu + \displaystyle\int_0^{2\pi} \kappa(s)^3 ds} = M \end{cases}$$

where $\mu,\ \nu$ are some constants.

Let us set the start point $s = 0$ at the maximal point of $\kappa(s)$ and n be a natural number. Each $\kappa(s)$ has a primitive period L/n, which is symmetric in $[0, 2\pi/n]$ at $s = \pi/n$, strictly decreasing in $(0, \pi/n)$ and strictly increasing in $(\pi/n, 2\pi/n)$. (We call this solution " *n-mode solution*".)

This problem is first considered by K. Watanabe [4–6] with the winding number $\omega = 1$. M. Murai - W. Matsumoto - S. Yotsutani [1–3] investigated the global structure of the Euler equation and shapes of its solutions.

We give the shapes of one of 2-mode solutions with the winding number $\omega = 1$ corresponding to amounts M in Figure 1. We give those of the other 2-mode solutions with $\omega = 1$ in Figure 2, which are always non-simple.

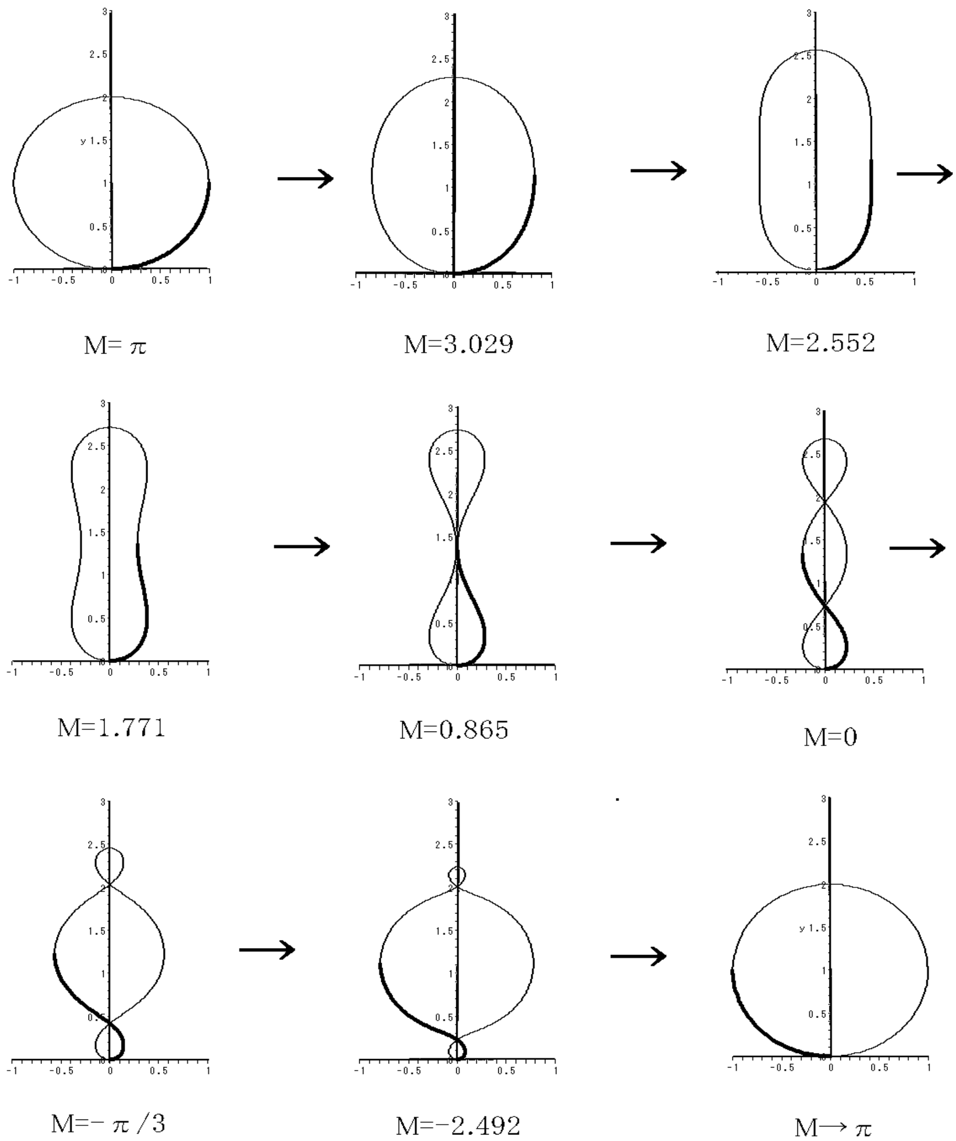


Figure 1: Curves by 2-mode solutions with $\omega = 1$

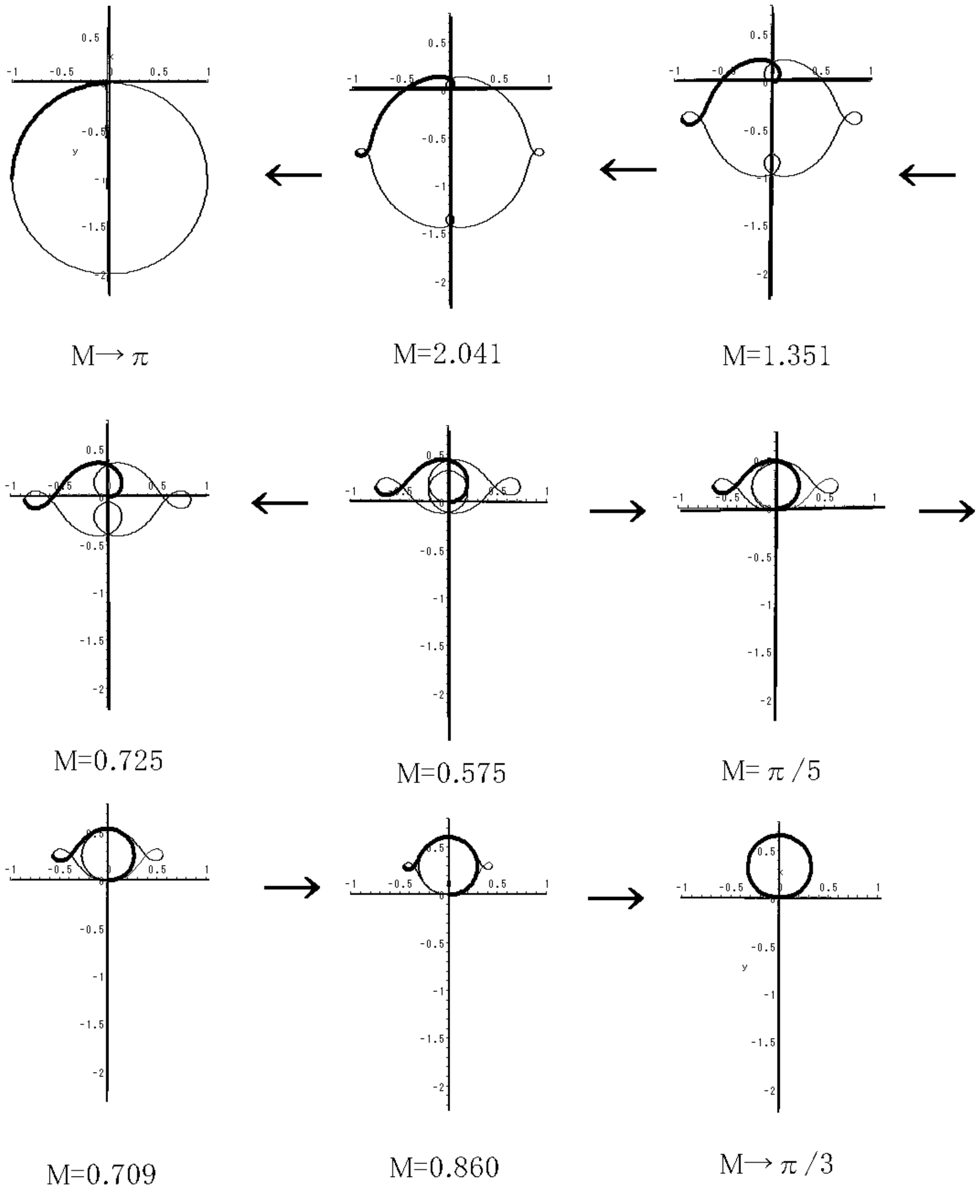
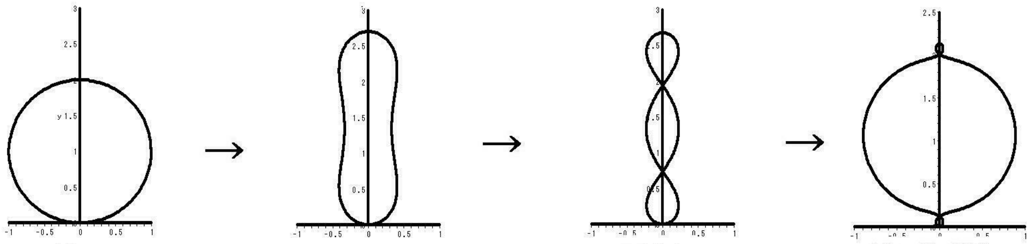


Figure 2: Curves by 2-mode solutions with $\omega = 1$

In this paper, we consider the the blow up phenomena of the curvature with the winding number $\omega \geq 1$. For example, in the case of the winding number $\omega = 1$ and mode $n = 2$, we obtain the following theorem.

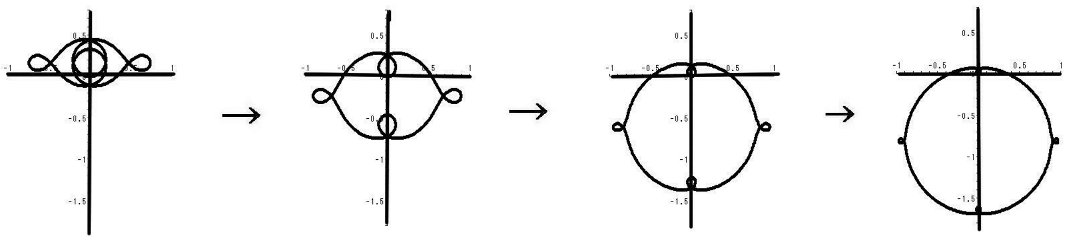
Theorem. Let $\omega = 1$ and $n = 2$. Then, there exists three types of blow up phenomena:

(i) $\displaystyle\lim_{M\downarrow -\pi} \kappa(s;M) = \begin{cases} \infty & s = 0, \pi, \\ -1 & s \in (0,\pi)\cup(\pi,2\pi) \quad \text{(uniformly in the wider sense).} \end{cases}$



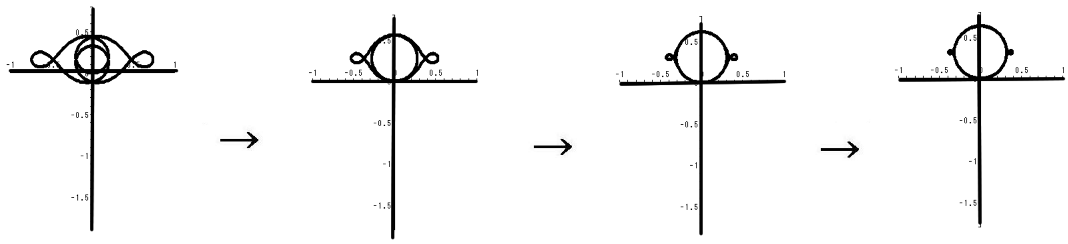
M=π M=1. 896 M=0 M=−2. 492

(ii) $\displaystyle\lim_{M\uparrow \pi} \kappa(s;M) = \begin{cases} \infty & s = 0, \pi, \\ 1 & s \in (0,\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2})\cup(\frac{3\pi}{2},2\pi), \\ -\infty & s = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{(uniformly in the wider sense).} \end{cases}$



M=0.575 M=1.054 M=1.920 M=2.490

(iii) $\displaystyle\lim_{M\uparrow \frac{\pi}{3}} \kappa(s;M) = \begin{cases} 3 & s \in [0,\frac{\pi}{2})\cup(\frac{\pi}{2},\frac{3\pi}{2})\cup(\frac{3\pi}{2},2\pi], \\ -\infty & s = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{(uniformly in the wider sense).} \end{cases}$



M=0.575 M=0.851 M=0.891 M=0.953

References

[1] W. Matsumoto, M. Murai, and S. Yotsutani, *By which kind of sound can one hear the shape of a drum?*, RIMS Koukyuroku **1315** (2003), 156–175.

[2] ———, *What have we learned on the problem: Can one hear the shape of a drum?*, Phase Space Analysis of Partial Differential Equations, vol. II, CENTRO DE RICERCA MATEMATICA ENNIO DE GIORGI, SCUOLA NORMALE SUPERIORE PISA (2005), 345–361.

[3] ———, *One can hear the shapes of some Non-Convex drums*, More Progress in Analysis Proceeding of the 5th International ISAAC Congress 2009. World Scientific Publishing. (2009), 863–872.

[4] K. Watanabe, *Plane domains which are spectrally determined*, Ann. Global Anal. Geom. **18** (2000), no. 5, 447–475.

[5] ———, *Plane domains which are spectrally determined. II*, J. Inequal. Appl. **7** (2002), no. 1, 25–47.

[6] ———, *Variational Methods in the Inverse Eigenvalue Problem of Laplace Operator*, Tokyo Institute University, 2004. doctor thesis.