Note on lower bounds of energy growth for solutions to wave equations

Shin-ichi Doi, Tatsuo Nishitani and Hideo Ueda
Department of Mathematics, Osaka University

International Conference the 26th MATSUYAMA Camp, January 8, 2011

1 Abstract

In this lecture we discuss about lower bounds of energy growth for solutions to

$$\partial_t^2 u - \sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(t,x)\partial_{x_j} u) = 0 \tag{1}$$

where $a_{ij}(t,x) = a_{ji}(t,x)$ are smooth with bounded derivatives of all orders such that

$$\begin{cases} a_{ij}(t,x) = \delta_{ij}, \quad |x| \geq R_1, \\ A^{-2}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j \leq A^2|\xi|^2, \ (t,x) \in \mathbb{R}^{1+n} \end{cases} \tag{2}$$

with some $R_1 > 0$, $A > 0$, that is, (1) is a *compact in space perturbation* of the wave equation $\partial_t^2 u - \Delta u = 0$.

In what follows we put

$$a(t,x,\xi) = \sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j.$$

Denote by $\mathcal{H}$ the Hilbert space which is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^n} \sum_{i=1}^{n} |\partial_{x_i} u|^2 dx = \int_{\mathbb{R}^n} |\nabla u|^2 dx.$$

Let $\mathcal{R}(t,0)$ be the solution operator defined by

$$C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n) \ni \begin{pmatrix} u(0,\cdot) \\ \partial_t u(0,\cdot) \end{pmatrix} \mapsto \begin{pmatrix} u(t,\cdot) \\ \partial_t u(t,\cdot) \end{pmatrix} \in C_0^\infty(\mathbb{R}^n) \times C_0^\infty(\mathbb{R}^n)$$

which extends uniquely to bounded operator in $\mathcal{H} \times L^2$. We first give a simple upper bound on the possible growth of $\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$;

Proposition 1 *We have*

$$\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} \le C\exp\Big(\frac{1}{2}\int_0^t \Big[\sup_{x,\xi}\frac{|\partial_t a(\tau,x,\xi)|}{a(\tau,x,\xi)}\Big]d\tau\Big)$$

with some $C>0$.

We now investigate lower bounds on $\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$. We assume that there is a bicharacteristic $(x(t),\xi(t))$ of $\sqrt{a(t,x,\xi)}$ or $-\sqrt{a(t,x,\xi)}$ with $\xi(t)\neq 0$;

$$\frac{dx}{dt} = \pm\frac{\partial}{\partial\xi}\sqrt{a(t,x,\xi)},\quad \frac{d\xi}{dt} = \mp\frac{\partial}{\partial x}\sqrt{a(t,x,\xi)} \tag{3}$$

such that

$$|x(t)|\le C^*,\quad |\xi(t)|\ge c^* \tag{4}$$

with some $C^*>0$, $c^*>0$ for $t\ge 0$. Then we have

Theorem 1 *Assume that there is a bicharacteristic verifying (4). Then there is a positive constant* C *such that*

$$\begin{aligned}\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)} &\ge C\exp\Big(\frac{1}{4}\int_0^t \frac{\partial_t a}{a}(\tau,x(\tau),\xi(\tau))d\tau\Big)\\ &\ge CA^{-1}\sqrt{|\xi(t)|/|\xi(0)|}.\end{aligned}$$

Remark: We can easily check that if $|x(t)|$ remains in a bounded set for $t\ge 0$ then we have

$$\int_0^t \frac{\partial_t a}{a}(\tau,x(\tau),\xi(\tau))d\tau = \log\frac{a(t,x(t),\xi(t))}{a(0,x(0),\xi(0))}$$

and hence

$$\int_0^t (\partial_t a/a)(\tau,x(\tau),\xi(\tau))d\tau\to\infty,\quad t\to\infty$$

is equivalent to $\lim_{t\to\infty}|\xi(t)|=\infty$. In particular, if $\xi(t)$ is periodic in t then Theorem 1 gives no information about energy growth.

We construct examples inspired by Colombini and Rauch [1] to which one can apply Theorem 1 to get lower bounds on $\|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$. Our construction works in all dimensions $n\ge 2$ though we present only the case $n=2$ for simplicity. Consider the wave equation

$$\partial_t^2 u - \sum_{i=1}^2 \partial_{x_i}(a(t,x)\partial_{x_i}u) = 0 \tag{5}$$

that is, $a_{12}=a_{21}=0$ and $a_{11}=a_{22}=a(t,x)$ which is smooth with bounded derivatives of all orders and

$$C^{-1}\le a(t,x)\le C,\ \ (t,x)\in\mathbb{R}^{1+2},\quad a(t,x)=1\quad\text{when}\quad |x|\ge 2 \tag{6}$$

with some $C>0$.

Theorem 2 *For any smooth non-negative bounded function $\delta(t)$ on $[0,\infty)$ and for any $\epsilon > 0$ there exists $a(t,x)$ satisfying (6) such that for the associate solution operator $\mathcal{R}$ to (5) we have*

$$C_1 \exp\Big(\int_0^t \delta(\tau)d\tau\Big) \le \|\mathcal{R}(t,0)\|_{\mathrm{Hom}(\mathcal{H}\times L^2)}$$
$$\le C_2 \exp\Big((2+\epsilon)\int_0^t \delta(\tau)d\tau\Big)$$

with some $C_i > 0$ independent of ϵ.

If we impose some conditions on $\delta(t)$ the upper bound of energy growth in Theorem 2 can be improved. Denote by H^1 the usual Sobolev space $H^1(\mathbb{R}^n)$ then

Theorem 3 *Let $\delta(t)$ be a smooth non-negative bounded function on $[0,\infty)$ such that $\delta'(t) \le 0$, $\delta''(t) \ge 0$. Then there exists $a(t,x)$ verifying (6) such that for the associate solution operator $\mathcal{R}(t,0)$ to (5) we have*

$$C_1 \exp\Big(\int_0^t \delta(s)ds\Big) \le \|\mathcal{R}(t,0)\|_{\mathrm{Hom}(H^1\times L^2;\mathcal{H}\times L^2)}$$
$$\le C_2 \exp\Big(\int_0^t \delta(s)ds\Big)$$

with some constants $C_i > 0$.

Actually our $a(t,x) = a(t,r,\theta)$ $(x = re^{i\theta})$ is given by

$$\sqrt{a(t,r,\theta)} = \exp\big(\chi(r)(r-1-2\delta(t)f(\theta - t - \pi/2)\big)$$

where $\chi(r) \in C_0^\infty(\mathbb{R})$, $0 \le \chi(r) \le 1$ which is zero near $r = 0$ and identically equal to 1 on a small neighborhood of $r = 1$. Here $f(t) \in C^\infty(\mathbb{R})$ is 2π periodic verifying

$$f(0) = 0, \;\; f'(0) = 1.$$

Let us take $\delta(t) = (1-\kappa)(1+t)^{-\kappa}$, $0 \le \kappa < 1$. Then Theorem 3 shows that there is an $a(t,x)$ satisfying (6) such that the solution operator $\mathcal{R}(t,0)$ verifies

$$C_1 e^{(1+t)^{1-\kappa}} \le \|\mathcal{R}(t,0)\|_{\mathrm{Hom}(H^1\times L^2;\mathcal{H}\times L^2)} \le C_2 e^{(1+t)^{1-\kappa}}.$$

If we choose $\delta(t) = m(1+t)^{-1}$, $m > 0$ then from Theorem 3 one can find an $a(t,x)$ with (6) such that the associate $\mathcal{R}(t,0)$ satisfies

$$C_1(1+t)^m \le \|\mathcal{R}(t,0)\|_{\mathrm{Hom}(H^1\times L^2;\mathcal{H}\times L^2)} \le C_2(1+t)^m.$$

References

[1] F.Colombini and J.Rauch, *Smooth localized parametric resonance for wave equations*, J. reine angew. Math. **616** (2008), 1-14.