The construction of a class of solutions to hyperbolic equations and applications to inverse problem

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Dedicated to 60th birthday of Professor Yoshinori Morimoto

1 Background

Let n be an integer with $n \geq 2$, $x = (x^1, \cdots, x^n) \in \Omega$ and Ω be a bounded domain in $\mathbb{R}^n$. For $A(x) = (A_j(x))_{1 \leq j \leq n} \in C^2(\Omega; \mathbf{R}^n)$, $q \in L^\infty(\Omega; \mathbf{C})$, we define the operator of second order

$$\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big) = \sum_{j=1}^{n} \Big(\frac{1}{i}\frac{\partial}{\partial x_j} + A_j(x)\Big)^2 + q(x).$$

Assume 0 is not an eigenvalue of $\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big) : H^2(\Omega) \cap H^1_0(\Omega) \to L^2(\Omega)$.

Boundary value problem (BVP)

$$\begin{cases} \mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = f & \text{on } \partial\Omega \end{cases}$$

For $f \in H^{\frac{1}{2}}(\partial\Omega)$, we have the unique solution $u \in H^1(\partial\Omega)$.

Dirichlet and Neumann map (DN map)

$$\begin{aligned} &\Lambda_{A,q} : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega), \quad \text{by} \\ &\Lambda_{A,q}(f) = \Big(\frac{\partial}{\partial\nu} + iA\cdot\nu\Big)u|_{\partial\Omega}, \quad \text{for } f \in H^{\frac{1}{2}}(\partial\Omega), \end{aligned}$$

where ν be the unit outer normal vector.

The inverse problem (Calderón problem)

Can we determine the potential terms dA and q from DN map $\Lambda_{A,q}$? This means the possibility of knowing the inside from the boudary measurement. Calderón [2] gave the fundamental result and approach for this problem. There are many results for this problem after Calderón's work.

we have many problems as (i) uniqueness, (ii) stability estimate (ill-posed problem), (iii) reconstruction method, (iv) experiment (numerical and so on). We can consider related problems for $\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big) = -\Delta_g$, where $g = g_{jk}(x)dx^j dx^k$ is the Riemannian metric.

We can know the importance of DN map from the fact that Green formula gives

$$\begin{aligned}&\Big(\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)u, v\Big)_{L^2(\Omega)} - \Big(u, \mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)v\Big)_{L^2(\Omega)}\\ &= \Big(u|_{\partial\Omega}, \Big(\frac{\partial}{\partial \nu} + iA\cdot\nu\Big)v|_{\partial\Omega}\Big)_{L^2(\partial\Omega)} - \Big(\Big(\frac{\partial}{\partial \nu} + iA\cdot\nu\Big)u|_{\partial\Omega}, v|_{\partial\Omega}\Big)_{L^2(\partial\Omega)}.\end{aligned}$$

Bukhgeim-Uhlmann [1] showed that Partial data of DN map implies the uniqueness of $q \in L^\infty(\Omega)$ $(A(x) = 0)$.

Key idea

We construct the solutions $u = u(x;h)$ to $(\Delta + q)u = 0$ by

$$u(x;h) = e^{\frac{1}{h}\Phi(x)}\Big(a_0(x) + hr(x;h)\Big),$$

where $h > 0$ is a small parameter (semi-classical parameter). The phase function $\Phi(x)$ is defined by

$$\Phi(x) = \Phi_0(x) = (a+ib)\cdot x = \sum_{j=1}^{n}(a_j + ib_j)x^j$$

for $a, b \in \mathbb{R}^n$ with $|a| = |b| = 1$ and $a \cdot b = 0$. This phase function is linear. The linear phase function is important for these results.

In fact, the direct calculation gives

$$\Delta\Big(e^{\frac{1}{h}(a+ib)\cdot x}\Big) = \frac{1}{h^2}\Big\{\big(|a|^2 - |b|^2\big) + 2ia\cdot b\Big\}e^{\frac{1}{h}(a+ib)\cdot x}.$$

For the phase function of complex valued
$\Phi(x) = \varphi(x) + i\psi(x)$

$$
\begin{aligned}
&e^{-\frac{1}{h}\Phi(x)}\mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)\Big(e^{\frac{1}{h}\Phi(x)}\big(a_0(x) + hr(x;h)\big)\Big) \\
&= \Big\{-\frac{1}{h^2}\Big(\sum_{j=1}^{n}\frac{\partial \Phi}{\partial x_j}\Big) + \frac{1}{h}\mathcal{P}_1\Big(x, \frac{\partial}{\partial x}\Big) + \mathcal{L}\Big(x, \frac{\partial}{\partial x}\Big)\Big\}\big(a_0(x) + hr(x;h)\big).
\end{aligned}
$$

In the case of the linear phase function, the operator $\mathcal{P}_1\Big(x, \frac{\partial}{\partial x}\Big)$ becomes of Cauchy-Riemann type ($\overline{\partial}$).

About the lower order term $r(x;h)$

The lower order term $r(x;h)$ is obtaind from the weighted L^2 estimate for

$$
\int_\Omega e^{\frac{2}{h}a\cdot x}|v|^2 dx
$$

for a semi-classical parameter $h > 0$. This is the Carleman estimate with the weight function $\varphi_0(x) = a \cdot x$. Carleman estimates (but delicate case) and Riesz theorem imply the construction pf the lower order term $r(x;h)$.

What is the role of this special solution?

We want to show that $\Lambda_{q_1} = \Lambda_{q_2}$ implies $q_1 = q_2$.

If $\Lambda_{q_1} = \Lambda_{q_2}$, we have

$$
\int_\Omega (q_1 - q_2)uv dx + (\text{small as } h \to 0) = 0.
$$

Set u and v as exponentially growing and decaying:

$$
\begin{aligned}
u &= e^{\frac{1}{h}(\varphi_0(x) + i\psi_0(x))}\big(a_0(x) + hr_0(x;h)\big) \\
v &= e^{\frac{1}{h}(-\varphi_0(x) + i\widetilde{\psi}_0(x))}\big(\widetilde{a}_0(x) + h\widetilde{r}_0(x;h)\big).
\end{aligned}
$$

$h \to 0$ implies

$$
\int_\Omega (q_1 - q_2)e^{\frac{i}{h}(\psi_0(x) + \widetilde{\psi}_0(x))}dx = 0,
$$

for $a_0(x) = \widetilde{a}_0(x) = 1$.
For the linear phase $\psi_0(x) + \widetilde{\psi}_0(x) = (b + \widetilde{b}) \cdot x$, we have the Fourier fransform of $q_1 - q_2$. The assumption $n \geq 3$ implies that b can be perturbed. So the analytic wave front set is calculated and support theorem is proved. This is the rough sketch of the proof of the final step.

2 Results on the constant metric case

Let Ω be a domain in $\mathbb{R}^n$ $(n \geq 2)$ with $\Omega^c = \mathbb{R}^n \backslash \Omega = \phi$ $x = (x^1, \cdots, x_n) \in \Omega$ and $x_0 = (x_0^1, \cdots, x_0^n) \in \Omega^c$. Let $G = (g_{jk})_{1 \leq j,k \leq n}$ be a real constant matrix with $G = G^t$. Assume $\det G \neq 0$. We have the inverse matrix $G^{-1} = (g^{jk})_{1 \leq j,k \leq n}$ We define the pseudo-metric by

$$g = \sum_{j,k=1}^{n} g_{jk} dx^j dx^k$$

and the pseudo-distance between x and x_0 by

$$||x - x_0||_G^2 = \sum_{j,k=1}^{n} g_{jk}(x^j - x_0^j)(x^k - x_0^k).$$

We define the operators of second order

$$P\Big(x, \frac{1}{i}\frac{\partial}{\partial x}\Big) = \sum_{j,k=1}^{n} g^{jk}(x)\Big(\frac{1}{i}\frac{\partial}{\partial x_j}\Big)\Big(\frac{1}{i}\frac{\partial}{\partial x_k}\Big) + (\text{lower order terms}).$$

The principal symbol $p_2(x,\xi)$ for $(x,\xi) \in T^*\Omega$ is defined by $p_2(x,\xi) = \sum_{j,k=1}^{n} g^{jk}(x)\xi_j\xi_k$. For $(x,\zeta) \in \Omega \times \mathbb{C}^n$, $p_2(x,\zeta) = \sum_{j,k=1}^{n} g^{jk}(x)\zeta_j\zeta_k$ is also defined. We shall construct a solution $\Phi(x) = \varphi(x) + i\psi(x)$ $(\varphi(x), \psi(x) \in C^2(\widetilde{\Omega}, \mathbb{R}))$ of complex valued to the eikonal equation

$$\begin{aligned} 0 =& p_2(x, \nabla\Phi(x)) = \sum_{j,k=1}^{n} g^{jk} \frac{\partial \Phi}{\partial x^j}\frac{\partial \Phi}{\partial x^k} \\ =& \sum_{j,k=1}^{n} g^{jk}\Big(\frac{\partial \varphi}{\partial x^j}\frac{\partial \varphi}{\partial x^k} - \frac{\partial \psi}{\partial x^j}\frac{\partial \psi}{\partial x^k}\Big) + 2i\sum_{j,k=1}^{n} g^{jk}\frac{\partial \varphi}{\partial x^j}\frac{\partial \psi}{\partial x^k}. \end{aligned}$$

First we set $\varphi(x)$. Second we choose $\psi(x)$.

For $(x,\xi) \in T^*\mathbb{R}^n = \Omega \times \mathbb{R}^n$ and the function $\varphi(x)$,
we define the symbols $a = a(x,\xi)$ and $b = b(x,\xi)$ as

$$a(x,\xi) = \sum_{j,k=1}^{n} g^{jk}\Big(\frac{\partial \varphi}{\partial x^j}\frac{\partial \varphi}{\partial x^k} - \xi_j\xi_k\Big), \quad b(x,\xi) = \sum_{j,k=1}^{n} g^{jk}\frac{\partial \varphi}{\partial x^j}\xi_k.$$

Poisson bracket between $a(x,\xi)$ and $b(x,\xi)$ is defined by

$$\{a,b\}(x,\xi) = \sum_{j=1}^{n}\Big(\frac{\partial a}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial a}{\partial x_j}\frac{\partial b}{\partial \xi_j}\Big).$$

So we shall want to find the function $\psi(x)$ of the solution to the system of nonlinear partial differential equations of first order

$$a(x, \nabla\psi) = b(x, \nabla\psi) = 0$$

for the function $\varphi(x)$.

We introduce a condition of the real part of the phase function $\varphi(x)$.

Definition [Limiting Carleman weight (LCW)]

The function $\varphi(x) \in C^2(\widetilde{\Omega}; \mathbf{R})$ is called a limiting Carleman weight on $\widetilde{\Omega}$ $(\subset \Omega)$ if and only if

$$\{a, b\}(x, \xi) = 0 \quad \text{on} \quad J,$$

where $J = \{(x, \xi) \in T^*\widetilde{\Omega} | \ \ a(x, \xi) = b(x, \xi) = 0\}$

If the set J is a manifold, this manifold J is involutive. This is the sufficient condition for the solvability of the system of the nonlinear equations of first order.

When $g_{jk} = \delta_j^k$ (flat Laplacian case), Poisson bracket is obtained as

$$\{a, b\}(x, \xi) = \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \Big(\frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_k} - \xi_j \xi_k \Big).$$

For the linear phase function $\varphi(x) = a \cdot x = \sum_{k=1}^{n} a_k x^k$, we have $\{a, b\}(x, \xi) = 0$ for $(x, \xi) \in T^*\Omega$.

Kenig-Sjostrand-Uhlmann [4]

They introduce the new phase function

$$\begin{aligned} \varphi(x) &= \log |x - x_0|, \\ \psi(x) &= \operatorname{dist}(\frac{x - x_0}{|x - x_0|}, \omega_0), \quad \omega_0 \in S^{n-1} \end{aligned}$$

They introduce the new weight function $\varphi(x) = \log |x - x_0|$ of log type. Thanks to the function $\varphi(x) = \log |x - x_0|$, the result in Calderón problem was improved.

Question

Why (How) can they find the new function $\varphi(x) = \log|x - x_0|$? We give a answer for this question.

Proposition

Let $f(t) \in C^2(\mathbf{R}\backslash 0; \mathbf{R})$. Set the function $\varphi(x)$ by

$$\varphi(x) = f(t)|_{t=|x-x_0|_G^2} \quad \text{(radial type function)}$$

Then (A) and (B) are equivalent:
(A) $\varphi(x)$ is a limiting Carleman weight.
(B) $f(t) = C_1 \log|t| + C_2$.
This means that the log type is the only choice of the limiting Carleman weight of radial type. In fact, this result is true not only for elliptic case but also for hyperbolic case as the wave equation.

References

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