Boltzmann Equation and Existence of Solutions

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Dedicated to the 60th birthday of
Professor Yoshinori Morimoto

Abstract

There is an extensive literature on the existence theory of solutions for the Boltzmann equation. However, as far as the Cauchy problem is concerned, it is noticed that the solutions developed so far are classified only into three categories from the view point of the limit behaviors at the spatial infinity, that is, x-periodic solutions (solutions on torus), solutions vanishing at spatial infinity (solution neafr vacuum), and solutions approaching to equilibrium at the spatial infinity (perturbative solution of equilibrium). In this note we show other solutions are possible which have more general spatial limit behaviors.

More precisely we will show that if the initial data is non-negative and belongs to a locally uniform Sobolev space with respect to x-variable and to a usual Sobolev space with weigh of Maxwell type decay with respect to v-variable, then the Cauchy problem has a non-negative unique solution belonging to the same function space, for arbitrarily large initial data and for both the cutoff and non-cutoff case.

This is a recent joint work with R. Alexandre, Y. Morimoto, C.-J. Xu, and T. Yang. [9].

1 Boltzmann Equation

When there is no external force, the Boltzmann equation takes the form

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \tag{1.1}$$

where $f = f(t, x, v) \geq 0$ is the density of gas particles having space position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time $t > 0$, while the right hand side Q is the Boltzmann bilinear collision operator given by

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B\left(v - v_*, \sigma\right) \Big(g(v'_*) f(v') - g(v_*) f(v) \Big) d\sigma dv_* \tag{1.2}$$

where

$$f(v') = f(t, x, v') \quad \text{etc.},$$

and

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

are the relations between the pre- and post- collisional velocities. $\sigma \in \mathbb{S}^2$ is the angular direction of collision.

In addition to the special bilinear structure of the collision operator, the *collision cross-section* $B(v-v_*,\sigma)$ varies with different physical assumptions on the particle interactions and it plays an important role in the well-posedness theory for the Boltzmann equation. Actually, it is a function of only $|v-v_*|$ and θ where

$$\cos\theta = \Big\langle \frac{v-v_*}{|v-v_*|},\sigma\Big\rangle, \quad 0\le\theta\le\frac{\pi}{2},$$

θ being the collision angle.

Two classical examples of B are, for 3D case;

- hard sphere gas: $B=b_0|v-v'|$, $\quad b_0>0$ being a constant
- inverse power law potential $r^{-(p-1)}$, $p\in(2,\infty)$, where r denotes the distance between two interacting molecules:

  $$B(v-v_*,\sigma)=b_0|v-v_*|^\gamma\theta^{-2-2s}, \quad b_0>0 \text{ being a constant,}$$

  with
  $$-3<\gamma=\frac{p-5}{p-1}<1, \quad 0<s=\frac{1}{p-1}<1\,.$$

As usual, the hard and soft potentials correspond to $p>5$ and $2<p<5$ respectively, while the Maxwellian potential corresponds to $p=5$.

The collision operator Q behaves quite differently between these two examples. For the hard sphere model, the cross section $B(v-v_*,\sigma)$ is a bounded function with respect to θ, and hence Q is a usual integral operator in v-variable.

On the other hand, the inverse power law potential results in a non-integrable singularity at $\theta=0$ which makes Q behave like a pseudo differential operator in v-variable, as pointed out by many authors, e.g. [2, 19, 22, 26], This had been an obstacle of long standing for the progress of the mathematical analysis of the Boltzmann equation.

To avoid this difficulty, Grad [14] introduced an assumption to cutoff this singularity by replacing an integrable one. This was a substantial step made in the study of the Boltzmann equation (1.1) and is now called Grad's angular cutoff assumption.

One of the main issues in the study of the Boltzmann equaityon is the existence theory of the solutions. Many authors have developed various methods for constructing local and global solutions for different situations. Among them, the Cauchy problem has been studied most extensively for both cutoff and non-cutoff cases. Let us give a brief review of the existence theories developed so far.

Cutoff approximation

1. L^∞-Theory: solution on torus (x-periodic) /near equilibrium
   Spectrum analysis+bootstrap argument
   Ukai ([24] '74, [25]'76), Nishida-Imai ([21], '76), Shizuta-Asano([23], '78)

2. L^1-Theory: Large amplitude solution.
   Renormalized solutions+H-theorem
   Diperna-Lions([13] '89), Hamdache ('92) $\cdots$

3. L^2-Theory: Solutions near equilibrium.
   Macro-micro decomposition $+L^2$energy method.
   Liu-Yang-Yu([20] '04), Guo([16] '04)$\cdots$ $\quad$ (L^∞ solution$\cdots$Guo [17])

Non-cutoff case

1. Local solutions in Sobolev space in $\mathbb{R}^3_x\times\mathbb{R}^3_v$:
   Solutions vanishing at x-infinity $\sim$ solutions near vacuum.
   R. Alexandre - Y.Morimoto - S.U. - C.-J.Xu - T.Yang ([3], 2010)

2. Global solutions approaching equilibrium at x-infinity.
   R. Alexandre - Y.Morimoto - S.U. - C.-J.Xu - T.Yang ([**?**, **?**], 2010)

3. Global x-periodic (on torus) solutions
   Gressman-Strain ([15],2010)

We now make an essential observation that all the solutions mentioned above satisfy one of the following three spatial behaviors at infinity;

1. x-periodic (on torus) solutions, [14, 15, 24]),

2. Solutions approaching an equilibrium state at x-infinity ([4, 5, 6, 7, 16, 20, 25]).

3. Solutions near vacuum (solutions vanishing at infinity,) [1, 3, 10, 13]).

Recall that the equilibrium state of the gas is described by the Maxwellian distribution. Without loss of generarity, we here fix it as a normalized Maxwellian

$$\mu(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}.$$

The solution approaching the equilibrium or perturbative solution of equilibrium means the solution having the form

$$f(t,x,v) = \mu(v) + \mu^{1/2}(v) g(t,x,v), \quad g \to 0 \ (|x| \to \infty).$$

μ satisfies $Q(\mu,\mu) = 0$ and hence is a stationary solution of the Boltzmann equation.

A natural question arises: Whether or not there exist any solutions behaving differently at x-infinity. The answer is yes. It is clear that the restriction of the limit behaviors comes from the function spaces used in the existence proof of solutions: If the proof is carried out in the Sobolev space on the torus, we obtain periodic solutions, while in the Sobolev space in the whole space, solutions are those vanishing at infinity, and perturbative solutions of equilibrium are obtained if the form $\mu(v) + \mu^{1/2}(v) g(t,x,v)$ with g belonging to the Sobolev space in the whole space is assumed. Thus it is expected that different function spaces give rise to solutions behaving differently at infinity. In this note we choose the uniformly local Sobolev space.

Uniformly local Sobolev space

Set

$$\partial^\alpha_\beta = \partial^\alpha_x \partial^\beta_v, \qquad \alpha, \beta \in \mathbb{N}^3$$

and let $\phi_1 = \phi_1(x)$ be a smooth cutoff function

$$\phi_1 \in C^\infty_0(\mathbb{R}^3), \quad 0 \le \phi_1(x) \le 1, \quad \phi_1(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

Set $k \in \mathbb{N}$.

The uniformly local Sobolev space we use in this note is defined by

$$\begin{aligned} H^k_{ul}(\mathbb{R}^6) = \{ g \mid & \|g\|^2_{H^k_{ul}(\mathbb{R}^6)} \\ &= \sum_{|\alpha+\beta| \le k} \sup_{a \in \mathbb{R}^3} \int_{\mathbb{R}^6} |\phi_1(x-a) \partial^\alpha_\beta g(x,v)|^2 dx dv < +\infty \}. \end{aligned} \tag{1.3}$$

This space could be defined also with the cutoff function $\phi_R(x) = \phi_1(x/R)$ for any $R > 0$, but the choice of R is not a matter. The proof is easy and is omitted. The uniformly local Sobolev space was first introduced by Kato in [18] as a space of functions of x variable, and was used to develop the local existence theory on the quasi-linear symmetric hyperbolic systems without specifying the limit behavior at infinity.

This space shares many important properties with the usual Sobolev space such as the Sobolev embedding. For example

$$H^k_{ul}(\mathbb{R}^3_x \times \mathbb{R}^3_v) \subset L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v) \ \text{ if } \ k > 3.$$

However, it is clear that $H^k_{ul}(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ imposes no limit property at x-infinity.

Actually, we shall focus on the solutions enjoying the weight of Maxwell type decay in v. More precisely, our space of initial data is, with $\langle v\rangle = (1+|v|^2)^{1/2}$,

$$\mathcal{E}^k_0(\mathbb{R}^6) = \Big\{ g \in \mathcal{D}'(\mathbb{R}^6_{x,v});\, \exists\, \rho_0 > 0 \ s.t.\ e^{\rho_0 <v>^2} g \in H^k_{ul}(\mathbb{R}^6_{x,v}) \Big\},$$

while the space of solutions will be, for $T > 0$,

$$\begin{aligned} \mathcal{E}^k([0,T]\times \mathbb{R}^6_{x,v}) \ &= \ \Big\{ f \in C^0([0,T]; \mathcal{D}'(\mathbb{R}^6_{x,v}));\, \exists\, \rho > 0 \\ &\qquad\qquad s.t.\ e^{\rho\langle v\rangle^2} f \in C^0([0,T];\ H^k_{ul}(\mathbb{R}^6_{x,v})) \Big\}. \end{aligned}$$

The method of proof developed here works for local existence theory. The global existence in the same solution spaces is a big open issue and is our current subject. Also the present method works for the Landau equation but since the extension is straightforward, the detail is omitted.

Cutoff and non-cutoff assumption

In this note we assume that B behaves like a non-cutoff principal part of the inverse power law potential case,

$$\begin{aligned} &B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos\theta), \\ &\quad \Phi(|v - v_*|) = \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma, \quad \gamma \in (-3, 1), \\ &\quad b(\cos\theta) \approx K\theta^{-2-2s} \ \text{ when } \ \theta \to 0+, \quad s \in (0,1), \end{aligned} \tag{1.4}$$

for some constant $K > 0$. It is know that this assumption captures essential features of the inverse power law potential case. $\gamma > 0$, $= 0$, and < 0 are for the hard, Maxwellian, and soft, potential cases, respectively.

For the cutoff case we replace b by a bounded function

$$b_\varepsilon(\cos\theta) = \begin{cases} b(\cos\theta) & \theta > \varepsilon, \\ b(\cos\varepsilon) & \theta < \varepsilon, \end{cases}$$

where ε is any small positive number.

In the sequel we mainly consider the non-cutoff case because the computation is almost the same for the cutoff case.

2 Main Result

Our existence result is stated as follows.

Theorem 2.1 *Assume the condition* (1.4) *on the cross section B with $0 < s < 1/2$, $\gamma > -3/2$ and $2s + \gamma < 1$. If the initial data f_0 is non-negative and belongs to the function space $\mathcal{E}^{k_0}_0(\mathbb{R}^6)$ for some $k_0 \in \mathbb{N}, k_0 \geq 4$, then, there exists $T_* > 0$ such that the Cauchy problem*

$$\begin{cases} f_t + v\cdot\nabla_x f = Q(f,f), \\ f|_{t=0} = f_0(x,v), \end{cases} \tag{2.5}$$

admits a non-negative unique solution in the function space $\mathcal{E}^{k_0}([0,T_*]\times\mathbb{R}^6)$.

Remark 2.1 *Notice that the space $H^k_{ul}(\mathbb{R}^6_{x,v})$ contains not only the spaces $H^k(\mathbb{R}^6_{x,v})$ and $H^k(\mathbb{T}^3_x \times \mathbb{R}^3_v)$ but also the space of functions having the form $\mu + \mu^{1/2}g$, as its subsets. If μ is a global Maxwellian, then we have well-known perturbative solutions of equilibrium as in [5, 6, 7, 8, 22]. But more general functions are included in $H^k_{ul}(\mathbb{R}^6)$; for example, functions having different limits at x- infinity like shock profile solutions which attain different equilibrium at the right and left infinity, almost periodic functions, and bounded functions behaving in more general way at x-infinity. Thus Theorem 1 extensively extends conventional spaces of admissible initial data.*

Strategy of proof
Instead of constructing the solutions of the Cuchy problem (2.5) directly, we solve its modified problem. More precisely, we set, for any $\kappa, \rho > 0$,

$$T_0 = \rho/(2\kappa),$$

and

$$\mu(t, v) = \mu_{\rho,\kappa}(t, v) = e^{-(\rho-\kappa t)(1+|v|^2)},$$

$$f = \mu(t, v)g,$$

$$\Gamma(g, g) = \Gamma_{t,\rho,\kappa}(g, g) = \mu(t, v)^{-1}Q(\mu(t, v)g,\ \mu(t, v)g)$$

for $t \in [0, T_0]$.

Then the Cauchy problem (2.5) is reduced to

$$\left\{ \begin{array}{l} g_t + v \cdot \nabla_x g \ + \kappa(1 + |v|^2)g = \Gamma(g, g), \\ g|_{t=0} = g_0. \end{array} \right. \tag{2.6}$$

The main motivation of this reduction is to create an extra term $\kappa(1 + |v|^2)g$ which is essential to control the weight loss cause by Q as will be explained below. Moreover, as a byproduct, the solution acquires a weight when integrated in t. Set $W_\ell = (1+|v|^2)^\ell$, $\ell \in \mathbb{N}^3$ and modify the space (2.7) as

$$\begin{aligned} H^{k,\ell}_{ul}(\mathbb{R}^6) = \{g \mid \|g\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} \\ = \sum_{|\alpha+\beta|\le k} \sup_{a\in\mathbb{R}^3} \int_{\mathbb{R}^6} |\phi_1(x-a)\partial^\alpha_\beta W_\ell g(x, v)|^2 dxdv < +\infty\}. \end{aligned} \tag{2.7}$$

Further, define

$$\begin{aligned} &\mathcal{M}^{k,\ell}(]0, T[\times\mathbb{R}^6) \\ &\quad = \{g \mid \|g\|_{\mathcal{M}^{k,\ell}(]0,T[\times\mathbb{R}^6)} = \int_0^T \|g(t)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} dt < +\infty\}. \end{aligned}$$

We will prove the

Theorem 2.2 *Assume that $0 < s < 1/2$, $\gamma > -3/2$ and $2s + \gamma < 1$. Let $\kappa, \rho > 0$ and let $g_0 \in H^{k,\ell}_{ul}(\mathbb{R}^6)$, $g_0 \ge 0$ for some $k \ge 4$ and $\ell \ge 3$. Then there exists $T_* \in]0, T_0]$ such that the Cauchy problem* (2.6) *admits a unique non-negative solution satisfying*

$$g \in C^0([0, T_*];\ H^{k,\ell}_{ul}(\mathbb{R}^6)) \bigcap\ \mathcal{M}^{k,\ell+1}(]0, T_*[\times\mathbb{R}^6))\,.$$

Remark 2.2 *That the solutions belong to $\mathcal{M}^{k,\ell+1}$ implies that they acquire the weight gain of order 1 appears if integrated in t, though the initial data is assumed only to enjoy the weight of order ℓ.*

The proof of Theorem 2.2 relies of course on the estimates of the non-linear collision operator Q.

3 Fundament estimates of Q

The most important recent progress on the non-cutoff case is the establishment of precise estimates of Q from above and below. They are summerized as follows, [1], [4], [9], [18], [29]. Set

$$
\begin{aligned}
&L^2 = L^2(\mathbb{R}^3_v),\\
&L^p_\alpha, H^s_\alpha : L^p \text{ and Sobolev space with the weight } (1+|v|^2)^{\alpha/2}.\\
&r^+ = \max(0, r), \quad r \in \mathbb{R}.
\end{aligned}
$$

Theorem 3.1 (upper bound of Q)
Let $0 < s < 1$, $\gamma > \max\{-3, -2s-3/2\}$, $m \in [s-1, s]$, $\alpha \in \mathbb{R}$. Then,

$$
\begin{aligned}
\Big|\Big(Q(f,g),h\Big)_{L^2}\Big| \le C\Big(\|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}} + \|f\|_{L^2}\Big)\\
\times \|g\|_{H^{\max\{s+m,(2s-1+\epsilon)^+\}}_{\alpha^+ + (\gamma+2s)^+}} \|h\|_{H^{s-m}_{-\alpha}} .
\end{aligned}
\tag{3.8}
$$

Theorem 3.2 (lower bound of Q)
Suppose that $0 < s < 1, \gamma \in \mathbb{R}, f \ge 0, \not\equiv 0,\ f \in L^1_2 \bigcap L\log L$.

$$
\begin{aligned}
&C_f \|\langle D_v\rangle^s \langle v\rangle^{\gamma/2} g\|^2_{L^2}\\
&\quad \le -(Q(f,g),g)_{L^2} + C\|f\|_{L^1_{\max(\gamma^+, 2-\gamma^+)}} \|g\|^2_{L^2_{\gamma^+/2}(\mathbb{R}^n)}
\end{aligned}
\tag{3.9}
$$

holds with a constant C_f depending only of $\|f\|_{L^1_2}$ and $\|f\|_{L\log L}$.

Remark 3.1 *(1) Theorem 3.1 implies that Q induces both the regularity loss of order $2s$ and weight loss of order $(2s+\gamma)^+$ with respect to v-variable, which implies that Q cannot be Lipschitz continuous operator so that the usual fixed point theorem fails for constructing local solutions.*

(2) Theorem 3.2 implies a coercivity of Q, or more precisely, $-Q(f,g)$ is a positive definite operator with respect to g if f is non-negative.

(3) Take $m = s$. Then these two theorem implies that $Q(g,f)$ behaves like a pseudo differential operator

$$
Q(f,g) \sim -C_f \langle D_v\rangle^{2s} g + \ \textit{lower order term},
$$

with $C_f > 0$.

4 Control of derivative and weight losses

The proof of Theorem 2.2 requires the control of the derivative and weight losses caused by Γ. The idea for controlling the weight loss was already stated. Here we present an idea for controlling the derivative loss. The estimate of Γ are derived from Theorems 3.1 and 3.2 by using the formula

$$
\begin{aligned}
\Gamma(f,g) &= Q(Mf,g) + \iint_{\mathbb{R}^3\times\mathbb{S}^2} B(M_* - M'_*) f'_* g' dv_* d\sigma\\
&= S_1 + S_2,
\end{aligned}
$$

where $M = \mu = \mu_{\rho,\kappa}(t), t \in [0, \rho/(2\kappa)]$. We start with S_1. Suppose $f \ge 0$. Then, thanks to Theorem 3.2, A_1 can be estimated as

$$
\begin{aligned}
(S_1, g)_{L^2(\mathbb{R}^3_v)} &\le -C_{Mf}\|\langle D_v\rangle^s\langle v\rangle^{\gamma/2} g\|^2_{L^2(\mathbb{R}^3_v)} + C\|Mf\|_{L^1_{\max(\gamma^+,2-\gamma^+)}} \|g\|^2_{L^2_{\gamma^+}}\\
&\le C\|Mf\|_{L^1_{\max(\gamma^+,2-\gamma^+)}} \|g\|^2_{L^2_{\gamma^+}} .
\end{aligned}
$$

This cancels the derivative loss in v of order s.

Actually, another derivative loss appears for the proof of Theorem 2.2 because we shall need estimates of Γ in Sobolev norms and thus the Sobolev embedding should be used. Notice that a Leibniz-like formula holds with respect to v:

$$\partial_v^\beta \Gamma(Mf,g) = \sum_{|\beta_1+\beta_2+\beta_3|=|\beta|} \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} g, \partial^{\beta_3} M),$$

where

$$\mathcal{T}(F,G,m) = \iint Bm_*(F'_*G' - F_*G)dv_*d\sigma.$$

Consider the case $|\beta| = 1$. Without loss of generality, we take $\beta = (1,0,0)$. Then,

$$\begin{aligned}\partial^\beta \Gamma(Mf,g) &= \mathcal{T}(f, \partial_{v_1} g, M) + \mathcal{T}(\partial_{v_1} f, g, M) + \mathcal{T}(f, g, \partial_{v_1} M) \\ &= E_1 + E_2 + E_3.\end{aligned}$$

Clearly $(E_1, \partial_{v_1} g)$ can be estimated as S_1 if $f \geq 0$. Theorem 3.2 cannot be used for E_2 because $\partial_{v_1} f$ cannot be non-negative. Instead, we shall use Theorem 3.1 with $m = s$;

$$\begin{aligned}|(E_2, \partial_{v_1} g)_{L^2(\mathbb{R}^3_v)}| &= |(Q(M\partial_{v_1} f, g), \partial_{v_1} g)| \\ &\leq C\Big(\|M\partial_{v_1} f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}} + \|M\partial_{v_1} f\|_{L^2}\Big)\|g\|_{H^{2s}_{(\gamma+2s)^+}} \|\partial\partial_{v_1} g\|_{L^2_v}.\end{aligned}$$

Therefore if $2s \leq 1$ (mild singularity), we have no derivative loss;

$$|(E_2, \partial\partial_{v_1} g)_{L^2(\mathbb{R}^3_v)}| \leq \|f\|_{L^2} \|g\|^2_{H^1_{(\gamma+2s)^+}(\mathbb{R}^3_v)},$$

while if $2s > 1$ (strong singularity), the derivative loss of order $2s - 1$ appears and cannot be controlled. The term E_3 has no problem.

The singularity in the term S_2 is killed by the term $M_* - M'_*$. Indeed, by the Taylor expansion we have

$$M'_* - M_* = (\nabla_v M)(v_*) \cdot (v'_* - v_*) + \frac{1}{2}\nabla_v^2 M(v_* + \eta)(v_* - v'_*)^{\otimes 2}.$$

Note that

$$v'_* - v_* = \frac{|v - v_*|}{2}(\sigma - (k\cdot\sigma)k) + \frac{|v - v_*|}{2}(k\cdot\sigma - 1)k,$$

where $k = (v - v_*)/|v - v_*|$. Since

$$\int B(|v - v_*|, k\cdot\sigma)(\sigma - (k\cdot\sigma)k)d\sigma = 0$$

by symmetry, and noting

$$k\cdot\sigma - 1 = \cos\theta - 1 \sim \theta^2 \quad (\theta \to 0),$$

we can cancel the singularity of order θ^{-1-2s} contained in B if $2s < 1$ and thus S_2 can be a finite quantity.

All the ideas presented here are fully used for establishing a priori estimates of local solutions. See [9] for a detail.

5 Strategy of local construction

The strategy for the proof of Theorem 2.2 is in the same spirit as in (ARMA 2010, [3]).

First, we construct the approximate solutions for the case where b is replaced by b_ε by a sequence of iterative linear equations. More precisely, we decompose Γ as

$$\begin{aligned}\Gamma(g.h) &= \Gamma^{\epsilon}(g,h)\\ &= \Gamma_\varepsilon^{t,+}(g,h) - \Gamma_\varepsilon^{t,-}(g,h)\end{aligned}$$

where

$$\begin{aligned}\Gamma_\varepsilon^{t,+}(g,h) &= \iint_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\,\sigma)\mu_*(t)\,g'_*h'dv_*d\sigma,\\ \Gamma_\varepsilon^{t,-}(g,h) &= hL_\varepsilon(g),\\ L_\varepsilon(g) &= \iint_{\mathbb{R}^3_{v_*}\times\mathbb{S}^2_\sigma} B_\varepsilon(v-v_*,\,\sigma)\mu(t,v_*)\,g_*dv_*d\sigma,\end{aligned}$$

and define a sequence of approximate solutions $\{g^n\}_{n\in\mathbb{N}}$ by

$$(5.10)\qquad \begin{cases} g^0 = g_0\,;\\ \partial_t g^{n+1} + v\cdot\nabla_x g^{n+1} + \kappa\langle|v|\rangle^2 g^{n+1} + L_\varepsilon(g^n)g^{n+1}\\ \qquad\qquad = \Gamma_\varepsilon^{t,+}(g^n,\,g^n),\\ g^{n+1}|_{t=0} = g_0.\end{cases}$$

Given a non-negative g^n and g_0, it is not hard to construct a unique non-negative g^{n+1} for any $n\in\mathbb{N}$. Moreover, it is easy to see that if

$$(5.11)\qquad \begin{aligned}&g_0\in H^{k,\ell}_{ul}(\mathbb{R}^6),\quad g_0\ge 0,\\ &g^n\in L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6)),\qquad g^n\ge 0,\end{aligned}$$

for any $T\in]0,T_0[$, then the solution g^{n+1} is determined in the function class

$$(5.12)\qquad g^{n+1}\in L^\infty(]0,T[;\ H^{k,\ell-\gamma^+}_{ul}(\mathbb{R}^6)),\qquad g^{n+1}\ge 0,$$

Actually the weight loss (5.12) can be recovered. We establish a uniform estimate for n of the approximates solutions by the energy integral method. Set

$$\begin{aligned}&Y = L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))\cap\mathcal{M}^{k,\ell+1}(]0,T[\times\mathbb{R}^6),\\ &||g||^2_Y = \|g\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))} + \kappa\|g\|^2_{\mathcal{M}^{k,\ell+1}(]0,T[\times\mathbb{R}^6)}.\end{aligned}$$

Lemma 5.1 *Assume that $-3/2<\gamma\le 1$ and let $k\ge 4, l\ge 0, \varepsilon>0$. Then, there exist positive numbers C_1, C_2 such that if $\rho>0$, $\kappa>0$ and if g_0 and g^n satisfy* (5.11) *with some $T\le T_0$, the function g^{n+1} of the solutions of* (5.10) *enjoy the properties*

$$\begin{aligned}&g^{n+1}\in Y\\ &||g^{n+1}||_Y\le e^{C_1K_nT}\left(\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} + \frac{C_2}{\kappa}||g^n||^4_{L^4(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))}\right),\end{aligned}$$

where K_n is a positive constant depending on $\|g^n\|_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))}$ and κ.

We are now ready to prove the convergence of $\{g^n\}_{n\in\mathbb{N}}$. Fix $\kappa>0$. Let $D_0>0$ and let

$$(5.13)\qquad g_0\in H^{k,\ell}_{ul}(\mathbb{R}^6),\quad \|g_0\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)}\le D_0$$

be given. Introduce an induction hypothesis

$$\|g^n\|_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))} \le 2D_0 \tag{5.14}$$

for some $T \in]0, T_0]$. Notice that the factor 2 can be any number > 1. (5.14) is true for $n = 0$ due to (5.13). Suppose that this is true for some $n > 0$. We shall determine T independent of n. A possible choice follows from Lemma 5.1 with

$$e^{C_1 K_0 T} = 2, \quad \frac{2^4 C_2}{\kappa} T D_0^2 = 1 \quad \text{where} \quad K_0 = \frac{1}{\kappa}(2D_0 + (1+\kappa)^2)$$

or

$$T = \min\left\{\frac{\log 2}{C_1 K_0},\ \frac{\kappa}{2^4 C_2 D_0^2}\right\}. \tag{5.15}$$

and the induction hypothesis (5.14) is fulfilled for $n+1$, and hence holds for all n.

For the convergence, set $w^n = g^n(t) - g^{n-1}(t)$, for which (5.10) leads to

$$\begin{cases} \partial_t w^{n+1} + v\cdot\nabla_x w^{n+1} + \kappa\langle|v|\rangle^2 w^{n+1} = \Gamma^{t,+}_\varepsilon(w^n,\ g^n) \\ \qquad\qquad +\Gamma^{t,+}_\varepsilon(g^{n-1},\ w^n) - \Gamma^{t,-}_\varepsilon(w^n,\ g^{n+1}) - \Gamma^{t,-}_\varepsilon(g^{n-1},\ w^{n+1}), \\ w^{n+1}|_{t=0} = 0. \end{cases}$$

By the same computation as in Lemma 5.1, we get

$$\begin{aligned} ||w^{n+1}||_Y^2 \le \frac{1}{2} C_2 e^{C_1 K_0 T}\frac{1}{\kappa} T\Big\{ & \|g^{n+1}\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))} + \|g^n\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))} \\ & + \|g^{n-1}\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))}\Big\} \|w^n\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))}, \end{aligned}$$

with the same constants C_1, C_2 and K_0 as above. Then, (5.15) give

$$||g^{n+1} - g^n||_Y^2 \le 2^4 C_2 D_0^2 \kappa^{-1} T \|g^n - g^{n-1}\|^2_{L^\infty(]0,T[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))}.$$

Finally, choose T smaller if necessary so that

$$2^4 C_2 D_0^2 \kappa^{-1} T \le \frac{1}{4}.$$

Then, we have proved that for any $n \ge 1$,

$$||g^{n+1} - g^n||_Y \le \frac{1}{2}\ ||g^n - g^{n-1}||_Y. \tag{5.16}$$

Consequently, $\{g^n\}$ is a convergence sequence in Y, and the limit

$$g^\varepsilon \in Y,$$

is therefore a non-negative solution of the Cauchy problem (2.6). The estimate (5.16) also implies the uniqueness of solutions.

In order to prove that g^ε exists uniformly for ε, we shall establish uniform a priori estimates for the Cauchy problem (2.6).

Theorem 5.1 *Assume that $0 < s < 1$, $\gamma > -3/2$, $\gamma + 2s < 1$. Let $g_0 \in H^{k,\ell}_{ul}(\mathbb{R}^6), g_0 \ge 0$ for some $k \ge 4$, $l \ge 3$. Then there exists $T_* \in]0, T_0]$ depending on $\|g_0\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)}$ but not on ε such that if for some $0 < T \le T_0$,*

$$g^\varepsilon \in C^0(]0,T];\ H^{k,\ell}_{ul}(\mathbb{R}^6)) \cap \mathcal{M}^{k,\ell+1}(]0,T[\times\mathbb{R}^6) \tag{5.17}$$

is a non-negative solution of the Cauchy problem (2.6) *and if $T_{**} = \min\{T,\ T_*\}$, then it holds that*

$$\|g^\varepsilon\|_{L^\infty(]0,T_{**}[;\ H^{k,\ell}_{ul}(\mathbb{R}^6))} \le 2\|g_0\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)}. \tag{5.18}$$

Remark 5.1 *The case $T_* \leq T$ gives a uniform estimate of local solutions on the fixed time interval $[0, T_*]$ while the case $T < T_*$ gives an a priori estimate on the existence time interval $[0, T]$ of local solutions. The latter is used for the continuation argument of local solutions.*

In the following, $\rho > 0$, $\kappa > 0$ are fixed. Furthermore, recall $T_0 = \rho/(2\kappa)$. We start with a solution g^ε subject to (5.17) for some $T \in]0, T_0]$. Put

$$h_\ell^{\alpha,\beta} = \phi_1(x-a)W_\ell \partial_\beta^\alpha g^\varepsilon \tag{5.19}$$

and take the L^2 inner product of $h_\ell^{\alpha,\beta}$ and the equation for $h_\ell^{\alpha,\beta}$ obtained by applying $\phi_1(x-a)W_\ell\partial_\beta^\alpha$ to (2.6). In the below $\| \ \|$ and $(\ , \ )$ will stand for the $L^2(\mathbb{R}^6)$ norm and inner product respectively unless otherwise stated. Then we have

$$\frac{1}{2}\frac{d}{dt}\|h_\ell^{\alpha,\beta}\|^2 + \kappa\|h_{\ell+1}^{\alpha,\beta}\|^2 = (\Xi, h_\ell^{\alpha,\beta}), \tag{5.20}$$

where

$$\begin{aligned}\Xi =&\phi_1(x-a)W_\ell\partial_\beta^\alpha\Gamma(g^\varepsilon, g^\varepsilon) - [\phi_1(x-a)W_\ell\partial_\beta^\alpha, v\cdot\nabla_x]g^\varepsilon \\ &- \kappa[\phi_1(x-a)W_\ell\partial_\beta^\alpha, \langle v\rangle^2]g^\varepsilon \\ =&\Xi_1 + \Xi_2 + \Xi_3.\end{aligned}$$

We shall derive the estimates

Lemma 5.2 *Assume that* $0 < s < 1/2, \gamma \geq -3/2, \gamma + 2s < 1$. *Then,*

$$|(\Xi_1, h_\ell^{\alpha,\beta})| \leq C\|g_\ell^\varepsilon\|^2_{H^{k,\ell}_{ul}} \sum_{|\alpha'+\beta'|\leq k}\|h_{\ell+1}^{\alpha',\beta'}\|,$$

$$|(\Xi_2 + \Xi_3, h_\ell^{\alpha,\beta})| \leq C(1+\kappa+\|g^\varepsilon\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)})\|g^\varepsilon\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)}\|h_{\ell+1}^{\alpha,\beta}\|.$$

The proof relies heavily on the ideas stated in Section 4.

Use this lemma to evaluate the right hand side of (5.20) to deduce

$$\begin{aligned}&\frac{d}{dt}\|h_\ell^{\alpha,\beta}(t)\|^2 + 2\kappa\|h_{\ell+1}^{\alpha,\beta}\|^2 \\ \leq& \frac{C}{\kappa}((1+\kappa)^2 + \|g^\varepsilon\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)})\|g^\varepsilon\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} + \frac{\kappa}{k^3}\sum_{|\alpha'+\beta'|\leq k}\|h_{\ell+1}^{\alpha',\beta'}\|^2,\end{aligned}$$

and after integrating over $]0, t[$,

$$\begin{aligned}&\|h_\ell^{\alpha,\beta}(t)\|^2 + \kappa\int_0^t \|h_{\ell+1}^{\alpha,\beta}(\tau)\|^2 d\tau \\ &\leq \|g^{\alpha,\beta}(0)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} + \frac{C}{\kappa}\int_0^t((1+\kappa)^2 + \|g^\varepsilon(\tau)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)})\|g^\varepsilon(\tau)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}d\tau \\ &\quad + \frac{\kappa}{k^3}\sum_{|\alpha'+\beta'|\leq k}\int_0^t\|h_{\ell+1}^{\alpha',\beta'}(\tau)\|^2 d\tau.\end{aligned}$$

Take the supremum with respect to $a \in \mathbb{R}^3$ (see (5.19)) and sum up over $|\alpha+\beta| \leq k$ to deduce that

$$\begin{aligned}&\|g^\varepsilon(t)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} + \kappa\|g^\varepsilon\|^2_{\mathcal{M}^{k,\ell+1}(]0,t[\times\mathbb{R}^6)} \\ &\leq \|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} + \frac{C}{\kappa}\int_0^t(1+\|g^\varepsilon(\tau)\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)})^2\|g^\varepsilon(\tau)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}d\tau.\end{aligned}$$

Then the Gronwall type inequality gives for $C_\kappa = C/\kappa$,

$$\|g^\varepsilon(t)\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} \le \frac{\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)} e^{C_\kappa t}}{1 - \big(e^{C_\kappa t} - 1\big)\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}}\,, \tag{5.21}$$

as long as the denominator remains positive. We choose $T_* > 0$ small enough such that

$$\frac{e^{C_\kappa T_*}}{1 - \big(e^{C_\kappa T_*} - 1\big)\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}} = 4.$$

Then

$$T_* = \frac{1}{C_\kappa} \log\Big(1 + \frac{3}{1 + 4\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}}\Big)$$

is independent of $\varepsilon > 0$, but depends on $\|g_0\|_{H^{k,\ell}_{ul}(\mathbb{R}^6)}$ and the constant C which depends on ρ, κ, k and l. Now we have (5.18) for $T_{**} = \min(T, T_*)$.

From (5.18) and (5.20), we get also, for $\kappa > 0$,

$$\kappa\|g^\varepsilon\|^2_{\mathcal{M}^{k,\ell+1}(]0,T_{**}[\times\mathbb{R}^6)} \le 2\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)}\Big(1 + 2CT_*(1 + 2\|g_0\|^2_{H^{k,\ell}_{ul}(\mathbb{R}^6)})\Big). \tag{5.22}$$

We have proved Theorem 5.1.

Combine (5.21)and (5.22). Then the compactness argument as in Section 4.4 of [3] applies, to conclude the existence part of Theorem 2.2. The uniqueness part comes from Theorem 4.1 of [8]. Now the main Theorem 2.1 is proved by the help of Theorem 2.2, in the same manner as in Section 4.5 of [3].

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