Singularities of solutions to the Cauchy problem for second-order hyperbolic operators with the coefficients of their principal parts depending only on the time variable

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1. Introduction

Let $x = (x_0, x'') = (x_0, x_1, \cdots, x_n) \in \mathbf{R}^{n+1}$, and denote by $\xi = (\xi_0, \xi'') = (\xi_0, \xi_1, \cdots, \xi_n) \in \mathbf{R}^{n+1}$ their dual variables. The x_0 variable plays the role of the time variable. We consider second-order hyperbolic operators with symbols

$$P(x,\xi) = p(x_0,\xi) + \sum_{j=0}^{n} b_j(x)\xi_j + c(x),$$

where

$$p(x_0,\xi) = \xi_0^2 + \sum_{|\alpha|=2,\alpha_0\le 1} a_\alpha(x_0)\xi^\alpha.$$

We assume that

(A) the $a_\alpha(x_0)$ are real analytic on $[0,\infty)$ and $b_j(x), c(x) \in C^\infty(\overline{\mathbf{R}^{n+1}_+})$ ($0 \le j \le n$).

Here $\mathbf{R}^{n+1}_+ = \{x \in \mathbf{R}^{n+1};\ x_0 > 0\}$. We consider the following Cauchy problem:

(CP)
$$\begin{cases} P(x,D)u(x) = f(x) & \text{in } (0,\infty)\times \mathbf{R}^n, \\ D_0^j u(x)|_{x_0=0} = u_j & \text{in } \mathbf{R}^n \quad (\ j=0,1), \end{cases}$$

where $f \in C([0,\infty); \mathcal{D}'(\mathbf{R}^n))$ and $u_j \in \mathcal{D}'(\mathbf{R}^n)$ ($j = 0, 1$). We may assume by coordinate transformation

$$a_\alpha(x_0) \equiv 0 \quad \text{if } |\alpha| = 2 \text{ and } \alpha_0 = 1.$$

So $P(x,\xi)$ can be written as follows:

$$P(x,\xi) = \xi_0^2 - a(x_0,\xi'') + b_0(x)\xi_0 + b(x,\xi'') + c(x),$$
$$a(x_0,\xi'') = \sum_{j,k=1}^{n} a_{j,k}(x_0)\xi_j\xi_k, \quad b(x,\xi'') = \sum_{j=1}^{n} b_j(x)\xi_j, \quad a_{j,k}(x_0) = a_{k,j}(x_0).$$

We assume the following conditions:

(H) $a(x_0,\xi'') \ge 0$ for $(x_0,\xi'') \in [0,\infty)\times\mathbf{R}^n$.

(F) $b(x,\xi'') \equiv 0$ in x for any $\xi'' \in V$, where $V = \{\xi'' \in \mathbf{R}^n;\ a(x_0,\xi'') \equiv 0 \text{ in } x_0\}$.

If (CP) is C^∞ well-posed, then it follows from the Lax-Mizohata theorem and results in [IP] that (H) and (F) must be satisfied. By (H) V is a vector subspace of $\mathbf{R}^n$. So we may assume, with $1 \le n' \le n$, that $V = \{\xi'' \in \mathbf{R}^n;\ \xi_1 = \cdots = \xi_{n'} = 0\}$, since the case $V = \mathbf{R}^n$ is trivial. Then by (F) we have

$$a(x_0,\xi'') \equiv a(x_0,\xi') \not\equiv 0 \quad \text{in } x_0 \text{ for } \xi' \ne 0, \quad b(x,\xi'') \equiv b(x,\xi'),$$

where $\xi' = (\xi_1, \cdots, \xi_{n'})$. From (A) we have the following:

(i) For $T > 0$ there is $k_T \in \mathbf{N}$ such that $\sum_{j=0}^{k_T} |\partial_{x_0}^j a(x_0,\xi')| \ne 0$ for $(x_0,\xi') \in [0,T] \times S^{n'-1}$, where $S^{n'-1}$ denotes the $(n'-1)$ dimensional unit sphere.

(ii) There are $r \in \mathbf{N}$, real analytic functions $\lambda_j(x_0)$ and $v_{j,k}(x_0)$ ($1 \le j \le r$, $1 \le k \le n'$) defined on $[0,\infty)$ such that $\lambda_j(x_0) \not\equiv 0$, $a(x_0,\xi') = \sum_{j=1}^r \lambda_j(x_0)\zeta_j(x_0,\xi')^2$, where $\zeta_j(x_0,\xi') = \sum_{k=1}^{n'} v_{j,k}(x_0)\xi_k$.

Let Ω be a neighborhood of $[0,\infty)$ in $\mathbf{C}$ such that the $a_{j,k}(x_0)$ can be extended analytically to Ω, and define $\mathcal{R}(\xi') = \{(\operatorname{Re}\lambda)_+;\ \lambda \in \Omega \text{ and } a(\lambda,\xi') = 0\}$ for $\xi' \in \mathbf{R}^{n'} \setminus \{0\}$, where $a_+ = \max\{a, 0\}$. We assume

(L) For any $T > 0$ and $x'' \in \mathbf{R}^n$, there is $C > 0$ such that

$$\min_{t \in \mathcal{R}(\xi')} |x_0 - t|\, |b(x,\xi')| \le C\sqrt{a(x_0,\xi')} \quad \text{for } (x_0,\xi') \in [0,T] \times (\mathbb{R}^{n'} \setminus \{0\}),$$

where $\min_{t\in\mathcal{R}(\xi')} |x_0 - t| = 1$ if $\mathcal{R}(\xi') = \emptyset$.

(L) is a so-called Levi condition. Put

$$\Gamma(p(x_0,\cdot),\vartheta) = \{\xi \in \mathbf{R}^{n+1};\ \xi_0 > \sqrt{a(x_0,\xi')}\},$$
$$\Gamma^* = \{y \in \mathbf{R}^{n+1};\ y \cdot \xi \ge 0 \text{ for any } \xi \in \Gamma\},$$

where $\vartheta = (1, 0, \cdots, 0) \in \mathbf{R}^{n+1}$. We define for $x^0 \in \overline{\mathbf{R}^{n+1}_+}$

$$\begin{aligned} K^\pm_{x^0} = \{x(t);\ &\pm t \ge 0,\ \{x(t)\} \text{ is a Lipschitz continuous curve in } \overline{\mathbf{R}^{n+1}_+} \\ &\text{and } (d/dt)x(t) \in \Gamma(p(x_0(t),\cdot),\vartheta)^* \ a.e.\ t\} \\ (\subset \{x;\ &x_j = x_j^0\ (\ n'+1 \le j \le n)\}). \end{aligned}$$

Concerning C^∞ well-posedness we have the following

Theorem 1. (CP) has a unique solution $u \in C^2([0,\infty);\mathcal{D}'(\mathbf{R}^n))$. Let $x^0 \in \overline{\mathbf{R}^{n+1}_+}$. If u satisfies (CP) and

$$(\operatorname{supp} f \cup \{0\} \times (\operatorname{supp} u_0 \cup \operatorname{supp} u_1)) \cap K^-_{x^0} = \emptyset,$$

then $x^0 \notin \operatorname{supp} u$. Moreover, (CP) is C^∞ well-posed.

Remark. We assume that (H), (F) and (A) are satisfied. Moreover, we assume that the $a_{j,k}(x_0)$ are polynomials of x_0, for example, when $n' \geq 3$. Then (CP) is C^∞ well-posed if and only if (L) is satisfied.

For the proof of Theorem 1 we refer to [W].

2. Main results

Definition 1. Let $z^0 \equiv (x^0, \xi^0) \in \mathbf{R}_+^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\})$.

(i) The localization polynomial $p_{z^0}(X)$ at z^0 is defined by

$$p(z^0 + sX) = s^{r(z^0)}(p_{z^0}(X) + o(1)) \text{ as } s \to 0, \quad p_{z^0}(X) \not\equiv 0 \text{ in } X \in \mathbf{R}^{2n+2}$$

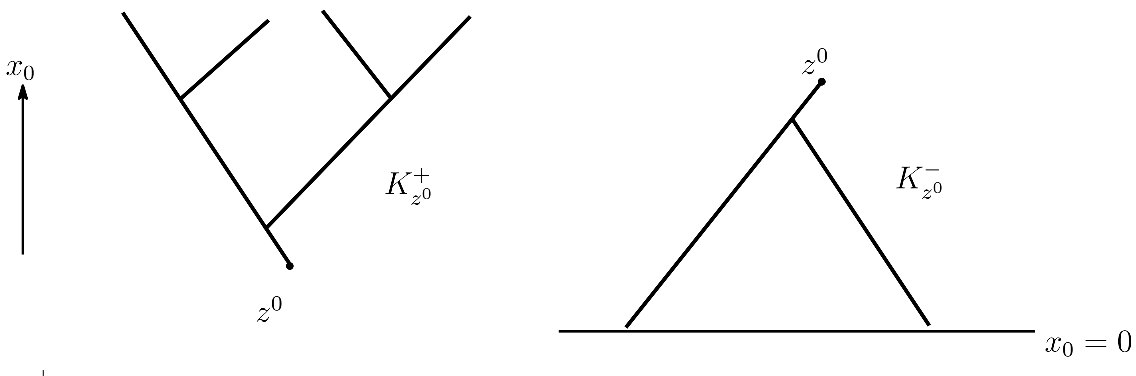
(ii) The generalized Hamilton flows $K_{z^0}^{\pm}$ are defined by

$$\begin{aligned} K_{z^0}^{\pm} \equiv & \{z(t);\ \pm t \geq 0,\ \{z(t)\} \text{ is a Lipschitz continuous curve in } T^*\mathbf{R}_+^{n+1} \setminus 0 \\ & \text{and } (d/dt)z(t) \in \Gamma(p_{z^0}, \widetilde{\vartheta})^\sigma \ a.e.\ t\}. \end{aligned}$$

Here $\widetilde{\vartheta} \equiv (0, \vartheta) \in \mathbf{R}^{2n+2}$, $\Gamma^\sigma = \{X \in \mathbf{R}^{2n+2};\ \sigma(Y, X) \geq 0 \text{ for any } Y \in \Gamma\}$ for $\Gamma \subset \mathbf{R}^{2n+2}$ and σ denotes the symplectic form on $T^*\mathbf{R}^{n+1}$.

Remark. $p_{z^0}(X)$ is hyperbolic w.r.t. $\widetilde{\vartheta}$.

Let $z^0 \equiv (x^0, \xi^0) \in \mathbf{R}_+^{n+1} \times (\mathbf{R}^{n+1} \setminus \{0\})$. If $\xi^{0\prime} = 0$, then $K_{z^0}^{\pm} = (K_{x^0}^{\pm} \cap \mathbf{R}_+^{n+1}) \times \{\xi^0\}$. If $p(x_0^0, \xi^{0\prime}) \neq 0$, then $K_{z^0}^{\pm} = \{z^0\}$. Moreover, $K_{z^0}^{\pm}$ are the broken null bicharacteristics of p in $T^*\mathbf{R}_+^{n+1} \setminus 0$ emanating from z^0 in the direction where $\pm x_0$ increase, if $\xi^{0\prime} \neq 0$ and $p(x_0^0, \xi^{0\prime}) = 0$. Assume that $\xi^{0\prime} \neq 0$ and $p(x_0^0, \xi^{0\prime}) = 0$.



$K_{z^0}^{\pm}$ branch at every double characteristic point. Each segment is a null bicharacteristics. Each null bicharacteristics satisfies the following:

$$\begin{cases} (d/dx_0)x''(x_0) = (\mp \nabla_{\xi'} \sqrt{a(x_0, \xi')}|_{\xi' = \xi^{0\prime}}, 0, \cdots, 0) \\ \xi_0(x_0) = \pm\sqrt{a(x_0, \xi^{0\prime})}, \quad \xi''(x_0) = \xi^{0\prime\prime} \end{cases}$$

By continuity $K^{\pm}_{z^0}$ can be defined as sets in $\overline{\mathbf{R}^{n+1}_+} \times (\mathbf{R}^{n+1} \setminus \{0\})$ for $z^0 \in \overline{\mathbf{R}^{n+1}_+} \times (\mathbf{R}^{n+1} \setminus \{0\})$.

Definition 2. Let $\delta > 0$ and $f \in C([0,\delta]; \mathcal{D}'(\mathbf{R}^n))$. $WF_0(f) \subset T^*\mathbf{R}^n \setminus 0$ can be defined as follows: We say that $z^{0\prime\prime} \equiv (x^{0\prime\prime}, \xi^{0\prime\prime}) \notin WF_0(f)$ if there are $\chi(x'', \xi'') \in S^0_{1,0}(\mathbf{R}^n)$, which is elliptic at $z^{0\prime\prime}$, and $\delta' > 0$ such that $\chi(x'', D'')f \in C([0,\delta']; H^\infty(\mathbf{R}^n))$.

Remrk. (i) The above definition is a variant of Chazarain's definition. (ii) $z^{0\prime\prime} \equiv (x^{0\prime\prime}, \xi^{0\prime\prime}) \notin WF_0(f)$ if and only if there are a neighborhood U'' of $x^{0\prime\prime}$, a conic neighborhood Γ'' of $\xi^{0\prime\prime}$ and $\delta' > 0$ such that for any $\varphi \in C^\infty_0(U'')$ there are $C_N > 0$ ($N \in \mathbf{N}$) satisfying

$$|\mathcal{F}_{x''}[\varphi(x'')f(x)](\xi'')| \le C_N \langle \xi'' \rangle^{-N}$$

for $N \in \mathbf{N}$, $x_0 \in [0,\delta']$ and $\xi'' \in \Gamma''$, where $\mathcal{F}_{x''}$ denotes the partial Fourier transformation with respect to x''.

Now we can state our main results.

Theorem 2. (I) Let $u \in \mathcal{D}'(\mathbf{R}^{n+1}_+)$ satisfy, with $\delta > 0$, $u \in C^2([0,\delta]; \mathcal{D}'(\mathbf{R}^n))$, and let $z^0 \equiv (x^0, \xi^0) \in WF(u)$, where $x^0_0 > 0$.

(i) When $0 < t < x^0_0$, $WF(u) \cap K^-_{z^0} \cap \{x_0 = t\} \neq \emptyset$ if $WF(Pu) \cap K^-_{z^0} \cap \{x_0 \ge t\} = \emptyset$.

(ii) When $t > x^0_0$, $WF(u) \cap K^+_{z^0} \cap \{x_0 = t\} \neq \emptyset$ if $WF(Pu) \cap K^+_{z^0} \cap \{x_0 \le t\} = \emptyset$.

(iii) If $WF(Pu) \cap K^-_{z^0} \cap \{x_0 > 0\} = \emptyset$, then

$$\begin{aligned}&(\bigcup_{j=0}^{1} WF((D^j_0 u)(0,x'')) \cup WF_0(Pu))\\ &\cap \{(x'',\xi'');\ (0,x'',\xi_0,\xi'') \in K^-_{z^0} \ \text{ for some } \xi_0 \in \mathbf{R}\} \neq \emptyset\end{aligned}$$

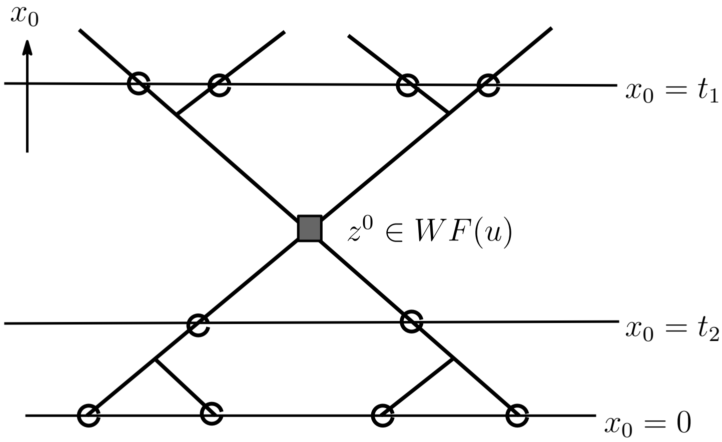
(II) (i) $\bigcup_{k=0}^2 WF_0(D^k_0 u) = (\bigcup_{j=0}^1 WF((D^j_0 u)(0,x'')) \cup WF_0(Pu))$.

(ii) Assume that the $a_{j,k}(x_0)$ can be extended to $\mathbf{R}$ so that $a_{j,k}(x_0) \in C^2(\mathbf{R})$ and $a(x_0, \xi') \ge 0$ and that $Pu \in C^\infty(\overline{\mathbf{R}^{n+1}_+})$, for simplicity. If $t > 0$ and $(x^{0\prime\prime}, \xi^{0\prime\prime}) \in \bigcup_{j=0}^1 WF((D^j_0 u)(0,x''))$, then

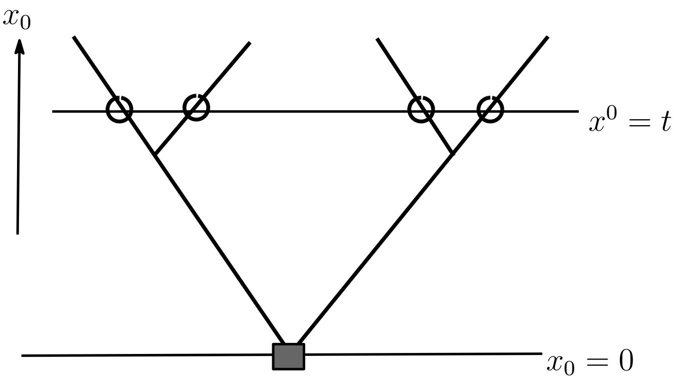
$$WF(u) \cap \{(x,\xi);\ x_0 = t \text{ and } (x,\xi) \in K^+_{(0,x^{0\prime\prime},\xi^0)} \text{ for some } \xi^0_0 \in \mathbf{R}\} \neq \emptyset.$$

Let us illustrate Theorem 2 with some figures. Assume that $Pu \in C^\infty(\overline{\mathbf{R}^{n+1}_+})$,

$K_{z^0}^+ \cup K_{z^0}^-$



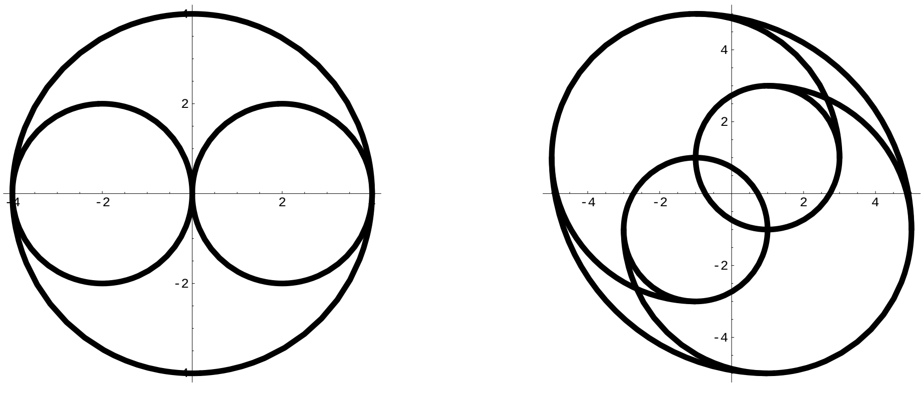
for simplicity, and that $z^0 \in WF(u)$. In the right figure the intersection $K_{z^0}^+ \cap \{x^0 = t_1\}$ consists of 4 points. Then Theorem 2 insists that at least one point of these 4 points in the intersection must belong to $WF(u)$. Similarly, at least one point of 2 points of the intersection $K_{z^0}^- \cap \{x^0 = t_2\}$ must belong to $WF(u)$ by Theorem 2. Moreover, at least one point of 4 points of $\{(x'', \xi'');\ (0, x'', \xi_0, \xi'') \in K_{z^0}^-$ for some $\xi_0 \in \mathbf{R}\}$ must belong to $\bigcup_{j=0}^1 WF((D_0^j u)(0, x'')$. Now we assume that $z^{0\prime\prime} \in \bigcup_{j=0}^1 WF((D_0^j u)(0, x''))$ and, for simplicity, $Pu \in C^\infty(\overline{\mathbf{R}_+^{n+1}})$. In the right figure the broken curves are equal to $\bigcup_\pm K^+_{(0, x^{0\prime\prime}, \pm\sqrt{a(0,\xi^{0\prime})}, \xi^{0\prime\prime})}$. The intersection of the broken curves and $\{x^0 = t\}$ consists of 4 points in this figure. Theorem 2 insists that at least one of these 4 points must belong to $WF(u)$.



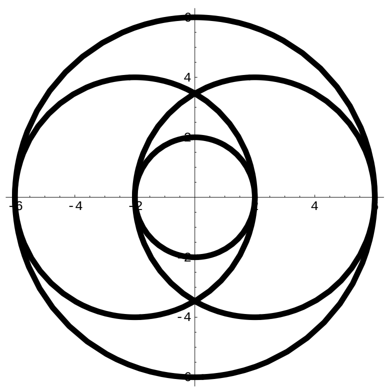
$z^{0\prime\prime} \in \cup_{j=0}^1 WF((D_0^j u)(0, x''))$

3. Examples

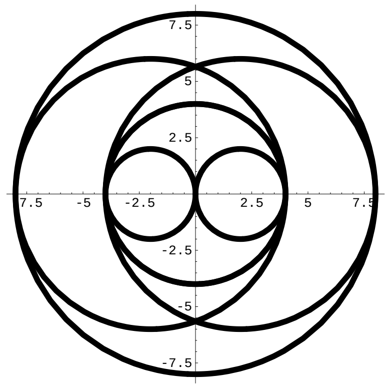
Example 1. Let $n = n' = 2$, $a(x_0, \xi'') = (-\xi_1 \sin x_0 + \xi_2 \cos x_0)^2$. Then $\bigcup_{\xi \neq 0} K^+_{(0,\xi)} \cap \{x_0 = t\}$ is the following:



$t = 2\pi$ $\qquad$ $t = 5/2\pi$



$t = 3\pi$



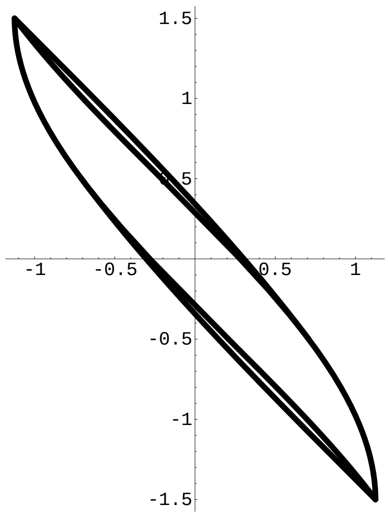
$t = 4\pi$

If $E(x)$ satisfies

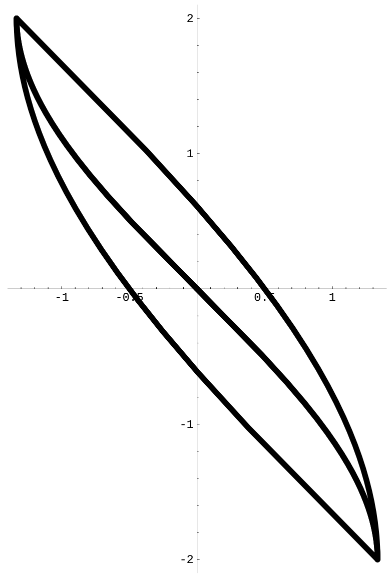
$$\begin{cases} P(x, D)E(x) = 0 & \text{in } \mathbf{R}^3_+, \\ E(0, x'') = 0, \quad (D_0E)(0, x'') = i\delta(x'') & \text{in } \mathbf{R}^2, \end{cases}$$

then $\operatorname{sing\,supp} E \subset \bigcup_{\xi \neq 0} K^+_{(0,\xi)}$. We could not prove the equality.

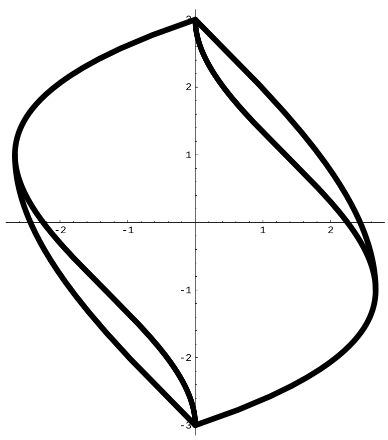
Example 2. Let $n = n' = 2$, $a(x_0, \xi'') = ((x_0^2 - 2x_0)\xi_1 + \xi_2)^2$ and $P(x, \xi) = p(x_0, \xi)$. Then $\operatorname{sing\,supp} E = \bigcup_{\xi \neq 0} K^+_{(0,\xi)}$ and $\bigcup_{\xi \neq 0} K^+_{(0,\xi)} \cap \{x_0 = t\}$ is as follows:



$t = 3/2$



$t = 2$



$t = 3$

Here, in order to prove the equality we have used the fact that $E(x_0, -x'') = E(x)$ and results on branching of singularities for operators with non involutive characteristics given by Hanges and Ivrii.

4. Outline of Proof of Theorem 2

In order to prove Theorem 2 (I) (i) or (ii) we use results given in [KW]. To prove Theorem 2 (I) (iii) and (II) we apply the same arguments as used in [KW]. Let $z^0 \in T^*\mathbf{R}^{n+1}_+$ satisfy $|\xi^0| = 1$, and choose $\vartheta^0 \in \Gamma(p_{z^0}, \widetilde{\vartheta})$ so that $\sigma(r(z^0), \vartheta^0) = 0$, where $r(x, \xi) = \sum_{j=0}^n \xi_j \dfrac{\partial}{\partial \xi_j}$. Put

$$\begin{aligned}
&\varphi(z; \kappa) = \tilde{\varphi}(z; \kappa)(1 + \tilde{\varphi}(z; \kappa)^2)^{-1/2}, \\
&\tilde{\varphi}(x, \xi; \kappa) = \sigma(\vartheta^0, (x - x^0, \xi/|\xi| - \xi^0)) + \kappa(|x - x^0|^2 + |\xi/|\xi| - \xi^0|^2), \\
&\Lambda(x, \xi) = B\Psi(\xi/h)(\varphi(x, \xi; \kappa) - \nu) \log\langle\xi\rangle_h + l \log(1 + \delta\langle\xi\rangle_h), \\
&P_\Lambda(x, D) = (e^{-\Lambda})(x, D)P(x, D)(e^{\Lambda})(x, D),
\end{aligned}$$

where $h \geq 1$, $\kappa, B, l, \nu > 0$, $\delta \in [0, 1]$, $\langle\xi\rangle_h = (h^2 + |\xi|^2)^{1/2}$ and $\Psi(\xi) \in S^0_{1,0}$ satisfies $\Psi(\xi) = 1$ for $|\xi| \geq 1$ and $\Psi(\xi) = 0$ if $|\xi| \leq 1/2$. We note that $-H_\varphi(z^0) \equiv (-(\nabla_\xi \varphi)(z^0), (\nabla_x \varphi)(z^0)) = \vartheta^0$. In order to prove Theorem 2 (i) it suffices to show the following microlocal Carleman type estimates, choosing c_0, c_1, h so that $0 < c_0 < x^0_0 < c_1$ and $h \gg 1$: For any $\kappa > 0$ there are $\nu_0 > 0$, $\chi_k(x, \xi) \in S^0_{1,0}$ ($k = 1, 2$) and $l_k \in \mathbf{R}$ ($k = 1, 2, 3$) such that the $\chi_k(z)$ are positively homogeneous of degree 0 for $|\xi| \geq 1$, $\chi_k(z) = 1$ near z^0, and "for any $\nu \in (0, \nu_0]$ there is $B_0 > 0$ such that ⌜for any $B \geq B_0$ there is $l_0 > 0$ such that for any $l \geq l_0$ there are $\delta_0 \in (0, 1]$ and $C > 0$ satisfying

$$\|\chi_1(x, D/h)v\|_{l_1} \leq C\{\|P_\Lambda(x, D)v\|_{l_2} + \|v\|_{l_1 - 1} + \|(1 - \chi_2(x, D/h))v\|_{l_3}\}$$

if $v \in C^\infty_0((c_0, c_1) \times \mathbf{R}^n)$ and $0 < \delta \leq \delta_0$.⌟ " Here $\| \cdot \|_l$ denotes the Sobolev norm of order l. So an essential part is to show the above estimates. We omit it as it is long. The proof of Theorem 2 will be given in a forthcoming paper.

References

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