REGULARITY OF SOLUTIONS FOR THE NON-CUTOFF BOLTZMANN EQUATION

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ABSTRACT. In this notes, we present some results about the regularity of solutions of non-cutoff Boltzmann equations. The mains results was obtained by a series collaboration works of R. Alexandre, Y. Morimoto, S. Ukai C.-J. Xu and T. Yang.

1. INTRODUCTION

About the hypo-elliptic operators, we recall the famous Hörmander's operators :

$$H = X_0 + \sum_{j=1}^{m} X_j^* X_j,$$

where $X_j = \sum_{k=1}^{d} a_{j,k}(x)\partial_j, j = 0, 1, \cdots, d; x \in \mathbb{R}^d$ and $X_j^* = -\sum_{k=1}^{d} \partial_j\big(a_{j,k}(x)\,\cdot\,\big)$. The so-called Hörmander's conditions is :

$$\text{Span}\big\{X_j, [X_{j_1}, \cdots, [\cdots[X_{j_{k-1}}, X_k]\cdots]\big\} = \mathbb{R}^d.$$

Some examples :

- Laplacian $\triangle_x = \partial^2_{x_1} + \cdots + \partial^2_{x_d}$ with $X_0 = 0, X_j = \partial_{x_j}$;
- Heat operator $\partial_t - \triangle_x$ with $X_0 = \partial_t, X_j = \partial_{x_j}$;

this is non degenerate case. For the degenerate case, we have :

- Kolmogorov operator $\partial_t + v \cdot \nabla_x - \triangle_v$ with $X_0 = \partial_t + v \cdot \nabla_x, X_j = \partial_{v_j}, t \in \mathbb{R}, x, v \in \mathbb{R}^n, d = 2n+1$;
- Simple model : $H = X_1^2 + X_2^2$ with $X_1 = \partial_{x_1}, X_2 = \partial_{x_2} + x_1^k\partial_{x_3}$, $d = 3$;

We have also a pseudo-differential operators case which play important rule for kinetic equations:

- Generalized Kolmogorov operators : $\partial_t + v \cdot \nabla_x + a(t,x,v)\Big(-\triangle_v\Big)^s, 0 < s < \infty.$

There exists different notions about regularizing of weak (or classical) solutions.

- **Hypoellipticity** : We say that the operators H is hypoelliptic, if for any $f \in C^\infty$, and any $u \in \mathscr{D}'$ satisfy the equation $Hu = f$ in the distribution sense, then $u \in C^\infty$. This definition can be generalized to non linear equation in the different weak sense. That means, any weak or classical solution is, in fact, more smooth (in the interior of domain, or up to the boundary). So that it is a generalized notion from elliptic equation.
- **Smoothing effect of Cauchy problem**: For the Cauchy problem of an evolution equation. If the solution is more smooth then the initial date, we say the smoothing effect of Cauchy problem. It is a generalized notion from heat equation.

- **Propagation of regularity (or singularity)** : Also for the Cauchy problem of an evolution equation. If the solution preserve the smooth property of the initial date, we say the Propagation of regularity (or singularity). It is a generalized notion from transport equation.

The questions of regularity of a solution can be asked in different function spaces, such as Sobolev space, analytic function and also Gevrey class. Also the optimal gain of regularity is a important problem. It is well known that the Hörmander's operators is hypoelliptic ib=n the Sobolev space. We study now the hypoellipticity of a class of kinetic equations.

We consider, in this notes, the Cauchy problem for Boltzmann equation,

$$f_t + v\cdot\nabla_x f = Q_B(f,f),\ f|_{t=0} = f_0(x,v). \tag{1}$$

The Boltzmann bilinear collision operator take the form

$$Q_B(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{g'_*f' - g_*f\right\} d\sigma dv_*\,,$$

where $f'_* = f(v'_*), f' = f(v'), f_* = (f(v_*), f = f(v)$, and for $\sigma\in\mathbb{S}^2$,

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\ v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma.$$

The cross-section behaves like

$$B(|v-v_*|,\cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \Big\langle \frac{v-v_*}{|v-v_*|}\,,\sigma\Big\rangle, \tag{2}$$

where for $\gamma > -3$ and $0 < s < 1$

$$\Phi(|z|) = |z|^\gamma, \quad b(\cos\theta) \approx \theta^{-2-2s},\ \theta \to 0^+ \tag{3}$$

where the singularity of $b(\cos\theta)$ near to $\theta = 0$ imply

$$\int_{\mathbb{S}^2} b(\cos\theta)d\sigma = +\infty,$$

which means **non cutoff**.

The fundamental Question : Existence of (global) solutions, uniqueness and regularity of solutions.

In this notes, we present the following smoothing effect results.

Theorem 1 (See [2, 3, 7])**.** *Let* $f\in L^\infty(]0,T[,H^5_\ell(\mathbb{R}^6_{x,v})), \forall \ell\in\mathbb{N}$, *be a non-negative solution of the inhomogeneous Boltzmann equation* (1). *Assume that*

$$f(t,x,v)\geq 0 \ \ and\ \ \|f(t,x,\cdot)\|_{L^1(\mathbb{R}^3_v)} > 0.$$

Then, $f\in C^\infty(]0,T[\times\mathbb{R}^3_x;\mathscr{S}(\mathbb{R}^3))$.

Where

$$H^k_\ell(\mathbb{R}^6_{x,v}) = \left\{ f\in\mathscr{S}'(\mathbb{R}^6_{x,v})\,;\, W^\ell f\in H^k(\mathbb{R}^6_{x,v})\right\}\,,$$

is the weighted Sobolev space and $W^\ell(v) = (1+|v|^2)^{\ell/2}$ is always the weight for v variable.

The results of above Theorem was improved by a series works, in [2], we study the linearized equations, in [3] we consider the non linear Boltzmann equation with a modified kinetic factors $\tilde\Phi(|z|) = (1+|z|^2)^{\gamma/2}$. Finally, we consider the full "true" case in [7] under the assumption (2) and (3) for $\gamma > -3$ and $0 < s < 1$. We compare the inhomogeneous Boltzmann equation with the generalized Kolmogorov equations for the regularity of solutions.

In [9, 10], we consider also the homogeneous Boltzmann equation and compare it with heat equation.

For the existence and uniqueness results, please see [4, 5, 6, 7, 8] and the notes of my collaborators in this proceeding.

2. COERCIVE AND UPPER BOUNDED ESTIMATES

The singularity assumption of collision operators (2) and (3) imply the coercivity estimate, for $g \geq 0$,

$$c_g \|f\|^2_{H^s_{\gamma/2}(\mathbb{R}^3_v)} \leq (-Q(g,f),f)_{L^2(\mathbb{R}^3_v)} + C\|g\|_{L^1_{\gamma+2s}(\mathbb{R}^3_v)} \|f\|^2_{L^2_{\gamma/2+s}(\mathbb{R}^3_v)},$$

where $c_g > 0$ depending on $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$.

This coercive estimate is proved, for Maxwellian case $\gamma = 0$ in [1]. We have a complete proof for all case in [3, 9].

We have also the following upper bound estimates :

$$\|Q(f,g)\|_{H^m_\alpha(\mathbb{R}^3_v)} \leq C\|f\|_{L^1_{\alpha^+ + \gamma+2s}(\mathbb{R}^3_v)} \|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}^3_v)}.$$

for mollified kinetic factor in [3], and "true" case in [7].

From the coercivity and upper bounded, we can think

$$-Q(g,f) \approx (-\triangle_v)^s + \text{lower order terms}.$$

Then the smoothing effect property is similar to heat equation in the spatial homogeneous case, and to Kolmogorov equation in inhomogeneous case.

By using the above coercive and upper bounded estimates, the pseudo-differential calculus imply that for any L^2 weak solution,

$$\Lambda^s_v f \in L^2, \quad \Lambda^{-s}_v Q(f,f) \in L^2.$$

3. HYPOELLITICITY

To prove the regularity of weak solution, we prove firstly the following hypoellipticity results.

Theorem 2 ([2]). *Assume that $0 \leq s' < 1$, $0 < s \leq 1$. Let $g \in H^{-s'}(\mathbb{R}^{2n+1})$, $\Lambda^s_v f \in L^2(\mathbb{R}^{2n+1})$ satisfies the kinetic equation :*

$$f_t + v \cdot \nabla_x f = g \in D'(\mathbb{R}^{2n+1}).$$

Then it follows that

$$\frac{\Lambda_x^{s(1-s')/(s+1)} f}{(1+|v|^2)^{ss'/2(s+1)}}, \quad \frac{\Lambda_t^{s(1-s')/(s+1)} f}{(1+|v|^2)^{s/2(s+1)}} \in L^2(\mathbb{R}^{2n+1}),$$

where $\Lambda_\bullet = (1-\triangle_\bullet)^{1/2}$.

The proof of this Theorem used the generalized uncertainty principle.

Let $\alpha \in \mathbb{N}^6, |\alpha| \leq 5$, we have

$$\partial_t(\partial^\alpha f) + v \cdot \nabla_x(\partial^\alpha f) = \partial^\alpha Q_B(f,f) - [v \cdot \nabla_x, \partial^\alpha] f.$$

Small gain of regularity

By using the classical Leibnitz formula,

$$\partial^\alpha Q_B(g,f) = \sum_{0 \leq \beta \leq \alpha} C^\alpha_\beta Q_B(\partial^\beta g, \partial^{\alpha-\beta} f),$$

the coercivity imply :

$$\Lambda^s_v \partial^\alpha f \in L^2_\ell,$$

then upper bounded estimate deduce

$$\Lambda_v^{-s}\partial^\alpha Q_B(f,f)\in L^2_\ell.$$

Using now the hypo-ellipticity of kinetic equation with $g=\partial^\alpha Q_B(f,f)-[v\cdot\partial_x,\ \partial^\alpha]f$, we have

$$\Lambda_x^{s_0}\partial^\alpha f\in L^2_\ell,\quad s_0=s(1-s)/(s+1)>0.$$

Now we need the following Leibnitz formula for fractional order derivation $0<\lambda<1$,

$$|D|^\lambda Q(fg)=Q\big(|D|^\lambda f,\ g\big)+Q(f,\,|D|^\lambda g)+C_\lambda\int_{\mathbb{R}^n}\frac{Q(f_h,\ g_h)}{|h|^{n+\lambda}}dh$$

where $f_h(y)=f(y)-f(y+h)$.

Gain of regularity of order 1 for x :

We then prove the high order regularity by iteration. If $ks_0<1$, then $\Lambda_v^s f,\Lambda_x^{ks_0}f\in H^5_\ell\Rightarrow\Lambda_x^{(k+1)s_0}f\in H^5_\ell$. We stop at $ks_0\geq 1$.

The **key point** in the proof is, for $\alpha\in\mathbb{N}^6,|\alpha|\leq 5$, we chose the test function

$$W^\ell\Lambda_x^{ks_0}\partial^\alpha S_N(D_x,D_v)f$$

where $S_N(D_x,D_v)$ is a mollified to make f belong to a higher order Sobolev space. We try to prove

$$\|W^\ell\Lambda_x^{(k+1)s_0}\partial^\alpha S_N(D_x,D_v)f\|_{L^2}\leq C$$

with C independent of N.

Gain of regularity of order 1 for v :

If $ks<1$, then by iteration up to $(k+1)s\geq 1$,

$$\Lambda_x^1 f,\Lambda_v^{ks}f\in H^5_\ell\ \Rightarrow\ \Lambda_v^{(k+1)s}f\in H^5_\ell\ \Rightarrow\ f\in H^6_\ell.$$

In this step, the key point is that the commutators

$$\|[W^\ell\Lambda_v^{ks}\partial^\alpha S_N(D_x,D_v),\ v\cdot\partial_x]f\|_{L^2}\leq C\|\Lambda_x^1 f\|_{H^5_\ell}.$$

This is the raison that we improve in the first step the regularity up to order 1 for x.

In the above 2 step, the commutators of $Q(\cdot,\cdot)$ with the operators $W^\ell\Lambda^\lambda\partial^\alpha S_N(D_x,D_v)$ require the so-called non linear microlocal analysis, or say Fourier analysis for non linear operators.

REFERENCES

[1] R. Alexandre, L. Desvillettes, C. Villani, B. Wennberg, Entropy dissipation and long-range interactions, *Arch. Ration. Mech. Anal.* **152** (2000), 327-355

[2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, T. Yang, Uncertainty principle and regularity for Boltzmann equation, *J. Funct. Anal.* **255** (8) (2008), 2013-2066

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Regularizing effect and local existence for non-cutoff Boltzmann equation, *Arch. Rational Mech. Anal.*,**198** (2010), 39-123.

[4] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Global existence and full regularity of the Boltzmann equation without angular cutoff, to appear in *Comm. Math. Phys .*, http://hal.archives-ouvertes.fr/hal-00439227/fr/

[5] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential, to appear in *J. Funct. Anal.*, http://hal.archives-ouvertes.fr/hal-00496950/fr/

[6] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Boltzmann equation without angular cutoff in the whole space: II, global existence for hard potential, to appear in *Analysis and Applications*, http://hal.archives-ouvertes.fr/hal-00477662/fr/

[7] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T.Yang, Boltzmann equation without angular cutoff in the whole space: III, Qualitative properties of solutions, Preprint HAL, http://hal.archives-ouvertes.fr/hal-00510633/fr/.

[8] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T.Yang, Bounded Solutions of the Boltzmann equation in the whole space, to appear in *Kinetic and Related Models*. Preprint HAL, http://hal.archives-ouvertes.fr/docs/00/52/99/40/PDF/AMUXY-KRM-revised-2.pdf

[9] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang, Smoothing effect of weak solutions for spatially homogeneous Boltzmann equation without angular cutoff, work in progress.

[10] Morimoto Y., Ukai S., Xu C.-J., Yang T.: Regularity of solutions to the spatially homogeneous Boltzmann equation without angular cutoff. *Discrete and Continuous Dynamical Systems - Series A* **24**, (2009) 187–212.