HYPOCOERCIVITY IN KINETIC THEORY AND THE CONVERGENCE TO THE EQUILIBRIUM

BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

R. ALEXANDRE, Y. MORIMOTO, S. UKAI, C.-J. XU, AND T. YANG

Dedicated to the 60th Birthday of Professor Yoshinori MORIMOTO

Abstract. It is known that the coupling of some degenerate dissipative operator and a conservative operator gives full dissipation and convergence to the equilibrium. Based on the estimates on the collision operators for the Boltzmann equation without angular cutoff, we obtain the convergence rate estimates on solutions to the equilibrium for both soft and hard potentials. Our approach combines the compensating function method introduced by Kawashima for the Boltzmann equation and the energy method.

1. introduction

The hypocoercivity theory which is closely related to, but is different from, the hypoellipticity theory has become one of the main focuses in the study of problems from mathematical physics. The main feature of this theory is that the coupling of a degenerate diffusion operator and a conservative operator may give the dissipation in all variables, and the convergence to the equilibrium state which lies in a proper subspace of the kernel of the diffusion operator. Breakthroughs have been made and substantial results have been obtained recently, especially by Villani and his collaborators, on problems in bounded domains or a torus. However, there are still many challenging problems remained unsolved.

We will focus on problems on the hypocoercivity theory for kinetic equations in the whole space and try to obtain the optimal convergence rates of solutions to the equilibrium states. One of the main differences between problems in a bounded domain and those in the whole space is that only algebraic convergence rates are expected in the whole space rather than (almost) exponential decay in the bounded domain.

Many kinetic equations and systems have the main structure

$$C(f) = D(f),$$

where C is a conservative operator and D is a degenerate dissipative operator usually on the velocity variables. The hypocoercivity theory is about the coupling of these two operators, which leads to a full dissipation in all variables and the convergence to the equilibrium in large time. Roughly speaking, the hypoellipticity issue is related to the fact that the interaction of the "non-dissipative" first-order operator B with the "dissipative", but not elliptic, part of an operator L in the form of

$$L = -\sum_{j=1}^{m} A_j^2 + B, \quad with \quad m < n = dim,$$

2000 *Mathematics Subject Classification.* 35A05, 35B65, 35D10, 35H20, 76P05, 84C40.

Key words and phrases. Boltzmann equation, non-cutoff cross sections, global existence, soft potential.

produces regularization in the missing directions so that the time evolution equation

$$f_t + Lf = 0,$$

presents some of the typical features of a parabolic equation. On the other hand, the hypocoercivity theory is about the full dissipative large time behavior of the solution to the equilibrium state in terms of convergence rates in all variables, which comes from the interaction of the conservative operator and the degenerate diffusive operator. The hypocoercivity theory has been studied and developed recently by many researchers, cf. the papers by Villani and the references therein. However, most of the results obtained so far are about the problems in a torus or in a bounded domain where (almost) exponential decay to the equilibrium can be obtained. For problems in the whole space or in an exterior domain, the convergence rate is no longer (almost) exponential but algebraic. Based on the previous work, we will focus on the perturbative problems around some equilibrium states, in particular, in the setting of non-angular cutoff.

The hypocoercivity theory has been investigated extensively for physical models which include the Boltzmann equation, oscillator chains, Fokker-Planck equation, etc. And some elegant theorems on the (multiple) commutators in the spirit of Hörmander's celebrated regularity theorem for hypoellitícity phenomena have been established. Besides the study on the problems in a bounded domain or in a torus, detailed studies on the Boltzmann equation in the whole space were also carried out which include the well-posedness theory in a new function space with less regularity assumption on the initial data, the optimal convergene rates of the solutions and their spatial derivatives in time to the equilibrium, and the effect of the external force on the convergence rate analysis.

2. Main results

For non-equilibrium gas, Boltzmann in 1872 derived a time evolution equation for a scalar function

$$f = f(t, x, v) \quad t \in \mathbb{R}^+, x \in \mathbb{R}^3, v \in \mathbb{R}^3,$$

which stands for the probability (number) density function of gas particles having position x and velocity v at time t satisfying:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f),$$

where Q, *the collision operator* describes the binary collision of molecules given by

$$Q(f, f) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \theta)(f' f'_* - f f_*) dv_* d\omega.$$

Here, $f = f(t, x, v)$, $f' = f(t, x, v')$, $f'_* = f(t, x, v'_*)$, $f_* = f(t, x, v_*)$ and $v' = v-((v-v_*)\cdot\omega)\omega$, $v'_* = v_* + ((v - v_*) \cdot \omega)\omega$, come from the Conservation of momentum and energy, that is

$$v + v_* = v' + v'_*, \quad |v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2.$$

There are two classical models:

• Hard sphere gas:

$$B(v - v_*, \theta) = q_0 |v - v_*| |\cos\theta|,$$

• Potential of inverse power with $U \sim r^{-\rho}$:

$$B(v - v_*, \theta) \sim |v - v_*|^\gamma |\theta|^{-2-2s} b_0(\theta),$$
$$\gamma = 1 - \frac{4}{\rho}, \quad s = \frac{1}{\rho},$$

where $b_0(\theta) \geq 0$ does not vanish near $\theta = 0$. The interaction potential is called a hard potential if $\rho > 4$, Maxwellian molecule if $\rho = 4$ and a soft potential if $1 < \rho < 4$.

Let us first review the works on convergence rates with Grad's angular cutoff assumption:

• Perturbation(denoted by u) around a global Maxwellian.

For hard potential, the following results are obtain:

(a) : Bounded Domain $\Rightarrow u = O(e^{-\sigma t})$ $(\exists \sigma > 0)$:
- $\mathbb{T}_x$ (Ukai);
- Bounded Domain with Boundary Conditions (Giraud, Asano, Shizuta, $\cdots$).

(b) : Unbounded Domain $\Rightarrow u = O(t^{-\sigma})$ $(\exists \sigma > 0)$:
- $\mathbb{R}_x$ (Ukai, Nishida-Imai);
- Exterior Domain (Ukai - Asano, $\cdots$);
- Cauchy problem in $L^2 \cap L^\infty_\beta$ (Ukai-Y., '06).

Note that the spatial derivatives of the solution in L^2 norm have the following properties:

– : The l-th order space derivative of the solution u decays as $0(t^{-\sigma_{q,l}})$ for L^q perturbation.

– : However, the velocity derivative $\partial_v^\ell u$ does not decay faster than $0(t^{-\sigma_{q,0}})$

Here,

$$\sigma_{q,l} = \frac{3}{2}(\frac{1}{q} - \frac{1}{2}) + \frac{l}{2}$$

On the other hand, imposing smallness conditions on the space derivatives leads to the almost exponential decay in torus even for soft potentials:

(1) : Strain-Guo '05: Consider the case $\mathbb{T}^3$ with the cutoff soft potential and let $\ell \geq 4$. For any k, if

$$a_k = \|u_0\|_{H^{\ell+k}_{x,v}}$$

is sufficiently small, then

$$\text{(AED)} \qquad \|u(t)\|_{H^\ell_{x,v}} \leq C a_k (1 + \frac{t}{k})^{-k}$$

holds for all $t \geq 0$. Here, it is required that

$$a_k \to 0 \quad (k \to \infty).$$

(2) : Desvillettes-Villani '03: Suppose that there exists a smooth global solution satisfying

$$u(t) \in BC^0([0, \infty); H^\ell_{x,v})$$

for sufficiently large $\ell > k$. Then, (AED) also holds.

However, the situation is quite different from the one obtained by Strain-Guo:

Here, the smallness condition on u_0 is not assumed and hence the solution may be large, but the existence of such smooth large solutions is a big open problem at the present moment.

For the case without angular cutoff, we obtain the following result on covergence rates to equilibrium.

Theorem. (AMUXY, 2010) Let $0 < s < 1$ and $f = \mu + \mu^{1/2} g$ be a global solution with initial datum $f_0 = \mu + \mu^{1/2} g_0$. We have the following two cases:

1) Let $\gamma+2s\geq 0$, $N\geq 6$, $\ell>3/2+2s+\gamma$. There exists $\varepsilon_0>0$ such that if $\|g_0\|^2_{L^1(\mathbb{R}^3_x;L^2(\mathbb{R}^3_v))}+\|g_0\|^2_{H^N_\ell(\mathbb{R}^6)}\leq\varepsilon_0<<1$, we have

$$\|g(t)\|^2_{L^2(\mathbb{R}^6)}=\|\mathbf{P}g(t)\|^2_{L^2(\mathbb{R}^6)}+\|(\mathbf{I}-\mathbf{P})g(t)\|^2_{L^2(\mathbb{R}^6)}\lesssim(1+t)^{-3/2},$$

$$\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\mathbf{P}g(t)\|^2_{L^2(\mathbb{R}^6)}+\sum_{|\alpha|\leq N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})g(t)\|^2_{L^2(\mathbb{R}^6)}\lesssim(1+t)^{-5/2}.$$

2) Let $\max\{-3,-\frac{3}{2}-2s\}<\gamma\leq-2s, N\geq 6,\ell\geq N+1$. There exists $\varepsilon_0>0$ such that if $\|g_0\|^2_{\tilde{\mathcal{H}}^N_\ell(\mathbb{R}^6)}\leq\varepsilon_0<<1$, we have

$$\sup_{x\in\mathbb{R}^3}\|g(t)\|^2_{H^{N-3}(\mathbb{R}^3_v)}\lesssim(1+t)^{-1}.$$

Remarks: 1. Note that the convergence rate in the case $\gamma+2s\geq 0$ is optimal because it is the same as those for the linearized equation. The proof is based on the combination of the L^p-L^q estimate on the solution operator to the linearized equation, and the energy method. The L^p-L^q estimate can be obtained by the compensating function introduced by Kawashima, or the spectrum estimate obtained by Pao.

2. The case when $\gamma+2s<0$ corresponds to the soft potential in the cutoff case with $\gamma<0$, and the optimal convergence rate is not known even for cutoff potentials. So the proof here is simply based on energy method.

In the following, we sketch some ideas of the proof.

I. $\gamma+2s\geq 0$: Compensating function + Energy method

Firstly, consider a linear equation

$$\begin{cases}(\partial_t+v\cdot\nabla_x+L)h=g,\\ \\ h\,|_{t=0}=h_0,\end{cases}$$

where L is the linearized collision operator.

Compensating function (Kawashima, 1990):

$S(\omega)$ is callled a compensating function if it satisfies the following three properties (Kawashima):

(i) $S(\cdot)$ is C^∞ on $\mathbf{S}^2$ taking values in the space of bounded linear operators on $L^2(\mathbf{R}^3)$, and $S(-\omega)=-S(\omega)$ for all $\omega\in\mathbf{S}^2$.

(ii) $iS(\omega)$ is self-adjoint on $L^2(\mathbf{R}^3)$ for all $\omega\in\mathbf{S}^2$.

(iii) There exists a constan $c_0>0$ such that for all $f\in L^2(\mathbf{R}^3)$ and $\omega\in\mathbf{S}^2$,

$$\mathcal{R}\langle S(\omega)(v\cdot\omega)f,f\rangle+\langle Lf,f\rangle\geq c_0(|\mathbf{P}f|_2^2+|(\mathbf{I}-\mathbf{P})f|^2_{\mathcal{D}}).\qquad(A)$$

Construction of the compensating function by Kawashima can be briefly described as follows:

Let

$$\mathcal{W}=\mathrm{span}\{e_j|j=1,2,\cdots,13\}.$$

Here, the orthonormal set of functions e_j is given by

$$e_1 = \mu^{\frac{1}{2}}, \quad e_{i+1} = v_i\mu^{\frac{1}{2}}, \quad i = 1,2,3, \quad e_5 = \frac{1}{\sqrt{6}}(|v|^2 - 3)\mu^{\frac{1}{2}},$$

and

$$\begin{aligned} e_{j+4} &= \sum_{i=1}^{3} \frac{c_{ji}}{\sqrt{2}}(v_i^2 - 1)\mu^{\frac{1}{2}}, j = 2,3 \\ e_8 &= v_1v_3\mu^{\frac{1}{2}}, \quad e_9 = v_2v_3\mu^{\frac{1}{2}}, \quad e_{10} = v_3v_1\mu^{\frac{1}{2}}, \\ e_{i+10} &= \frac{1}{\sqrt{10}}(|v|^2 - 5)v_i\mu^{\frac{1}{2}}, \quad i = 1,2,3, \end{aligned}$$

where the constant vectors $c_i = (c_{i1}, c_{i2}, c_{i3})$, $i = 2,3$ together with $c_1 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ form an orthonormal basis of $\mathbb{R}^3$.

Let $\mathbf{P}_0$ be the orthogonal projection from $L^2(\mathbb{R}^3_v)$ onto $\mathcal{W}$,

$$\mathbf{P}_0 g = \sum_{k=1}^{13}(g, e_k)_{L^2(\mathbb{R}^3_v)}e_k.$$

Set $W_k = \langle f, e_k\rangle$, $k = 1,2,\cdots,13$, and $W = [W_1,...,W_{13}]^T$, $W_I = [W_1,...,W_5]^T$, and $W_{II} = [W_6,...,W_{13}]^T$. Then w

$$\partial_t W + \sum_j V^j\partial_{x_j}W + \overline{L}W = \overline{h} + R,$$

where V^j $(j = 1,2,3)$ and $\overline{L}$ are the symmetric matrices defined by

$$\overline{L} = \{(\mathcal{L}e_l, e_k)_{L^2(\mathbb{R}^3_v)}\}_{k,l=1}^{13}, \quad V^j = \{(v_je_k, e_l)_{L^2(\mathbb{R}^3_v)}\}_{k,l=1}^{13},$$

and $\overline{h} = [(h,e_1)_{L^2(\mathbb{R}^3_v)}, ..., (h,e_{13})_{L^2(\mathbb{R}^3_v)}]^T$. Here R denotes the remaining term which contains the factor $(\mathbf{I} - \mathbf{P}_0)g$.

$$V(\xi) = \sum_{j=1}^{3} V^j\xi_j = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

with

$$V_{11}(\xi) = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ \xi_1 & 0 & 0 & 0 & a_1\xi_1 \\ \xi_2 & 0 & 0 & 0 & a_1\xi_2 \\ \xi_3 & 0 & 0 & 0 & a_1\xi_3 \\ 0 & a_1\xi_1 & a_1\xi_2 & a_1\xi_3 & 0 \end{pmatrix},$$

and

$$V_{21}(\xi) = V_{12}(\xi)^T = \begin{pmatrix} 0 & a_{21}\xi_1 & a_{22}\xi_2 & a_{23}\xi_3 & 0 \\ 0 & a_{31}\xi_1 & a_{32}\xi_2 & a_{33}\xi_3 & 0 \\ 0 & \xi_2 & \xi_1 & 0 & 0 \\ 0 & 0 & \xi_3 & \xi_2 & 0 \\ 0 & \xi_3 & 0 & \xi_1 & 0 \\ 0 & 0 & 0 & 0 & a_4\xi_1 \\ 0 & 0 & 0 & 0 & a_4\xi_2 \\ 0 & 0 & 0 & 0 & a_4\xi_3 \end{pmatrix},$$

where $a_1 = \sqrt{\frac{2}{3}}, a_{kj} = \sqrt{2}c_{kj}$, $k = 2,3$, $j = 1,2,3$, and $a_4 = \sqrt{\frac{3}{5}}$.

By setting

$$R(\xi) = \sum_{j=1}^{3} R^j \xi_j = \begin{pmatrix} \alpha \tilde{R}_{11} & V_{12} \\ -V_{21} & 0 \end{pmatrix},$$

with

$$\tilde{R}_{11} = \begin{pmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ -\xi_1 & 0 & 0 & 0 & 0 \\ -\xi_2 & 0 & 0 & 0 & 0 \\ -\xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

it holds for some suitable α,

$$Re\langle R(\omega)V(\omega)W, W\rangle \geq c_1|W_I|^2 - c_2 \sum_{k=2}^{4} |W_{II}|^2.$$

Hence, for any given $\omega \in \mathbf{S}^2$, set $R(\omega) \equiv \{r_{ij}(\omega)\}_{i,j=1}^4$ and then

$$S(\omega)g \equiv \sum_{k,\ell=1}^{4} \lambda r_{k\ell}(\omega)(g, e_\ell)_{L^2(\mathbb{R}^3_v)} e_k,$$

is a compensating function with $0 < \lambda \ll 1$.

$L^p - L^q$ **estimate:**

Taking the Fourier transform in x of the linear equation yields

$$\hat{g}_t + i|\xi|(v\cdot\omega)\hat{g} + \mathcal{L}\hat{g} = \hat{h},$$

where $\omega = \frac{\xi}{|\xi|}$.

Then take the inner product with $((1+|\xi|^2) - i\kappa S(\omega))\hat{g}$ and use the properties of the compensating function, to get

$$\begin{aligned} ((1+|\xi|^2)||\hat{g}||^2_{L^2(\mathbb{R}^3_v)} &- \kappa|\xi|(iS(\omega)\hat{g},\hat{g})_{L^2(\mathbb{R}^3_v)})_t \\ &+ \delta_0((1+|\xi|^2)|||(\mathbf{I}-\mathbf{P})\hat{g}|||^2 + |\xi|^2||\mathbf{P}\hat{g}||^2_{L^2(\mathbb{R}^3_v)}) \\ &\leq C(1+|\xi|^2)Re(\hat{g},\hat{h})_{L^2(\mathbb{R}^3_v)}. \end{aligned}$$

This implies that

$$E(\hat{g})_t + \delta_0 \frac{|\xi|^2}{1+|\xi|^2} E(\hat{g}) \leq C||\hat{h}||^2_{L^2(\mathbb{R}^3_v)},$$

where

$$E(\hat{g}) = ||\hat{g}||^2_{L^2(\mathbb{R}^3_v)} - \kappa \frac{|\xi|}{1+|\xi|^2}(iS(\omega)\hat{g},\hat{g})_{L^2(\mathbb{R}^3_v)} \sim ||\hat{g}||^2_{L^2(\mathbb{R}^3_v)},$$

when κ is chosen to be small. And this estimate yields

$$||\hat{g}||^2_{L^2(\mathbb{R}^3_v)} \leq C\exp\{-\frac{\delta_0|\xi|^2 t}{1+|\xi|^2}\}||\hat{g}_0||^2_{L^2(\mathbb{R}^3_v)} + C\int_0^t \exp\{-\frac{\delta_0|\xi|^2(t-s)}{1+|\xi|^2}\}||\hat{h}||^2_{L^2(\mathbb{R}^3_v)}(s)ds.$$

And this gives the following lemma.

Lemma (Kawashima). Let $k \geq k_1 \geq 0$ and $N \geq 4$. Assume that

(i) $g_0 \in H^N(\mathbb{R}^6) \cap Z_q$,

(ii) $h \in C^0([0,\infty[; H^N \cap Z_q)$,

(iii) $\mathbf{P}h(t,x,v) = 0$ for all $(t,x,v) \in [0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$.

(iv) $g(t,x,v) \in C^0([0,\infty[; H^N(\mathbb{R}^6)) \cap C^1([0,\infty[; H^{N-1}(\mathbb{R}^6))$ is a solution of the linear equation.

Then we have

$$\|\nabla_x^k g\|^2_{L^2(\mathbb{R}^6)} \le C(1+t)^{-2\sigma_{q,m}}(\|\nabla_x^{k_1} g_0\|_{Z_q(\mathbb{R}^6)} + \|\nabla_x^k g_0\|_{L^2(\mathbb{R}^6)})^2$$
$$+\int_0^t (1+t-s)^{-2\sigma_{q,m}}(\|\nabla_x^{k_1} h\|_{Z_q(\mathbb{R}^6)} + \|\nabla_x^k h\|_{L^2(\mathbb{R}^6)})^2 ds,$$

for any integer $m = k - k_1 \ge 0$, where $q \in [1,2]$ and

$$\sigma_{q,m} = \frac{3}{2}\Big(\frac{1}{q} - \frac{1}{2}\Big) + \frac{m}{2}.$$

Energy estimate:

Recall from existence theory, when $N \ge 6$ and $l > 3/2 + 2s + \gamma$, we have

$$\frac{d}{dt}\mathcal{E} + D \le 0,$$

where $\mathcal{E} = \|g\|^2_{H^N_l(\mathbb{R}^6)}$ and $D = \|\nabla_x \mathbf{P} g\|^2_{H^{N-1}(\mathbb{R}^6)} + |||(\mathbf{I}-\mathbf{P})g|||^2_{\mathcal{B}^N_l(\mathbb{R}^6)}$.

In fact, we can also prove

$$\frac{d}{dt}\mathcal{E}_1 + D \le C\|\nabla_x \mathbf{P} g\|^2_{L^2_{x,v}(\mathbb{R}^6)},$$

where $\mathcal{E}_1 = \|\nabla_x \mathbf{P} g\|^2_{H^{N-1}(\mathbb{R}^6)} + \|(\mathbf{I}-\mathbf{P})g\|^2_{H^N_l(\mathbb{R}^6)}$.

Estimate on the collision operator:

Recall

$$\|\Gamma(f,g)\|_{L^2(\mathbb{R}^3_v)} \lesssim \|f\|_{L^2(\mathbb{R}^3_v)}\|g\|_{H^{2s}_{(\gamma+2s)^+}(\mathbb{R}^3_v)}.$$

Hence, by using the fact that $N \ge 6$ and $\ell > 3/2 + 2s + \gamma$, Sobolev imbedding implies

$$\begin{aligned}
\|\Gamma(g,g)\|^2_{L^2(\mathbb{R}^6_{x,v})} + \|\nabla_x \Gamma(g,g)\|^2_{L^2(\mathbb{R}^6_{x,v})} &\lesssim \mathcal{E}^2 \lesssim \mathcal{E}_1\mathcal{E} + \|\mathbf{P} g\|^4_{L^2(\mathbb{R}^6_{x,v})},\\
\|\Gamma(g,g)\|^2_{Z_1} &\lesssim \mathcal{E}_1\mathcal{E} + \|\mathbf{P} g\|^4_{L^2(\mathbb{R}^6_{x,v})}.
\end{aligned}$$

Combination of the above three estimates: Define

$$M(t) = \sup_{0\le s\le t}\{(1+s)^{\frac{5}{2}}\mathcal{E}_1(s)\}, \quad M_0(t) = \sup_{0\le s\le t}\{(1+s)^{\frac{3}{2}}\|g(s)\|^2_{L^2(\mathbb{R}^6_{x,v})}\}.$$

Then by the $L^p - L^q$ estimate, we have

$$\begin{aligned}
\|\nabla_x g(t)\|^2_{L^2(\mathbb{R}^6_{x,v})} &\lesssim (1+t)^{-\frac{5}{2}}(\|g_0\|^2_{Z_1(\mathbb{R}^6)} + \|\nabla_x g_0\|^2_{L^2(\mathbb{R}^6_{x,v})})\\
&\quad + \int_0^t (1+t-s)^{-\frac{5}{2}}(\|\Gamma(g,g)\|_{Z_1(\mathbb{R}^6)} + \|\nabla_x\Gamma(g,g)\|_{L^2(\mathbb{R}^6_{x,v})})^2 ds\\
&\lesssim \eta(1+t)^{-\frac{5}{2}} + \int_0^t (1+t-s)^{-\frac{5}{2}}(\mathcal{E}\mathcal{E}_1 + \|\mathbf{P} g\|^4_{L^2(\mathbb{R}^6_{x,v})})(s) ds\\
&\lesssim \eta(1+t)^{-\frac{5}{2}} + \delta M(t)\int_0^t (1+t-s)^{-\frac{5}{2}}(1+s)^{-\frac{5}{2}} ds\\
&\quad + M_0^2(t)\int_0^t (1+t-s)^{-\frac{5}{2}}(1+s)^{-3} ds\\
&\lesssim \eta(1+t)^{-\frac{5}{2}} + \delta(1+t)^{-\frac{5}{2}} M(t) + (1+t)^{-\frac{5}{2}} M_0^2(t),
\end{aligned}$$

where $\eta = \|g_0\|^2_{Z_1(\mathbb{R}^6)} + \|g_0\|^2_{H^N_\ell(\mathbb{R}^6)}$, and $\mathcal{E} < \delta$. Thus, we have

$$\begin{aligned}\mathcal{E}_1(t) &\leq \mathcal{E}_1(0)e^{-t} + \int_0^t e^{-(t-s)}\|\nabla_x g\|^2_{L^2(\mathbb{R}^6_{x,v})}(s)ds\\ &\lesssim \delta e^{-t} + \eta(1+t)^{-\frac{5}{2}} + \delta(1+t)^{-\frac{5}{2}}M(t) + (1+t)^{-\frac{5}{2}}M_0^2(t),\end{aligned}$$

that is,

$$M(t) \lesssim (\delta+\eta) + \delta M(t) + M_0^2(t).$$

By applying the $L^p - L^q$ estimate again, we have

$$\begin{aligned}\|g(t)\|^2_{L^2(\mathbb{R}^6_{x,v})} &\lesssim (1+t)^{-\frac{3}{2}}(\|g_0\|^2_{Z_1(\mathbb{R}^6)} + \|g_0\|^2_{L^2(\mathbb{R}^6_{x,v})})\\ &\quad + \int_0^t (1+t-s)^{-\frac{3}{2}}(\|\Gamma(g,g)\|_{Z_1(\mathbb{R}^6)} + \|\Gamma(g,g)\|_{L^2(\mathbb{R}^6_{x,v})})^2(s)ds\\ &\lesssim \eta(1+t)^{-\frac{3}{2}} + \int_0^t (1+t-s)^{-\frac{3}{2}}(\mathcal{E}\mathcal{E}_1 + \|\mathbf{P}g\|^4_{L^2(\mathbb{R}^6_{x,v})})(s)ds\\ &\lesssim \eta(1+t)^{-\frac{3}{2}} + \delta(1+t)^{-\frac{3}{2}}M(t) + (1+t)^{-\frac{3}{2}}M_0^2(t).\end{aligned}$$

Hence,

$$\begin{aligned}M_0(t) &\lesssim \eta + \delta M(t) + M_0^2(t)\\ &\lesssim (\eta+\delta) + M_0^2(t).\end{aligned}$$

By assumption, $\eta + \delta$ is small. The above estimate and the continuity argument give $M_0(t) \leq C_{\eta,\delta}$, and then $M(t) \leq \bar{C}_{\eta,\delta}$, where $C_{\eta,\delta}$ and $\bar{C}_{\eta,\delta}$ are two constants depending on η and δ only.

Then the convergence rates for the hard potential case stated in the theorem follow.

II. $\gamma + 2s < 0$: energy method

We first recall the following lemma.

Lemma (Deckelnick). Let $f(t) \in C^1([t_0,\infty))$ such that

$$f(t) \geq 0, \qquad A = \int_{t_0}^\infty f(t)dt < \infty$$

and

$$f'(t) \leq a(t)f(t), \qquad t \geq t_0.$$

If $a(t) \geq 0$ and $B = \int_{t_0}^\infty a(t)dt < \infty$, then

$$f(t) \leq \frac{(t_0 f(t_0)+1)\exp(A+B) - 1}{t}, \qquad t \geq t_0.$$

First of all, the basic energy estimate derived for the global existence gives

$$\frac{d}{dt}\mathcal{E}_{N,\ell} + c_0\mathcal{D}_{N,\ell} \leq 0,$$

where $c_0 > 0$ is a constant. Here,

$$\begin{aligned}\mathcal{E}_{N,\ell} &\sim \|\mathcal{A}\|^2_{H^N(\mathbb{R}^3)} + \|g_2\|^2_{\widetilde{\mathcal{H}}^N_\ell(\mathbb{R}^6)},\\ \mathcal{D}_{N,\ell} &= \|\nabla_x\mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + \|g_2\|^2_{\widetilde{\mathcal{B}}^N_\ell(\mathbb{R}^6)},\end{aligned}$$

and

$$\widetilde{\mathcal{B}}^{N}_{\ell}(\mathbb{R}^6) = \Big\{ g \in \mathcal{S}'(\mathbb{R}^6);\ \|g\|^2_{\widetilde{\mathcal{B}}^{N}_{\ell}(\mathbb{R}^6)} = \sum_{|\alpha|+|\beta|\le N} \int_{\mathbb{R}^3_x} |||\tilde{W}_{\ell-|\beta|}\,\partial^\alpha_\beta g(x,\,\cdot\,)|||^2_{\Phi_\gamma} dx < +\infty \Big\}.$$

We can also construct another functional $\bar{\mathcal{E}}_{N-1,\ell-1}$ that has the following property

$$\bar{\mathcal{E}}_{N-1,\ell-1} \sim \|\nabla_x \mathcal{A}\|^2_{H^{N-2}(\mathbb{R}^3)} + \|\nabla_x g_2\|^2_{\widetilde{\mathcal{H}}^{N-2}_{\ell-1}(\mathbb{R}^6)},$$

and

$$\frac{d}{dt}\bar{\mathcal{E}}_{N-1,\ell-1} + \eta_0 \bar{\mathcal{D}}_{N,\ell-1} \lesssim \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3_x)}\big(\|\nabla_x \mathcal{A}\|^2_{H^{N-2}(\mathbb{R}^3_x)} + \|g_2\|^2_{L^2(\mathbb{R}^6)}\big) \lesssim \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3_x)} \bar{\mathcal{E}}_{N-1,\ell-1}.$$

Since

$$\int_0^\infty (\bar{\mathcal{E}}_{N-1,\ell-1} + \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3_x)})dt < \infty,$$

By using the assumption that $\ell - 1 \ge N$, the lemma implies that

$$\bar{\mathcal{E}}_{N-1,\ell-1} \lesssim (1+t)^{-1}.$$

Finally, the references of this note can be found in the references in our recent paper:

R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T.Yang, *Boltzmann equation without angular cutoff in the whole space: III, Qualitative properties of solutions*, Preprint HAL, http://hal.archives-ouvertes.fr/hal-00510633/fr/.

R. Alexandre, IRENAV Research Institute, French Naval Academy Brest-Lanvéoc 29290, France
and
Department of Mathematics, Shanghai Jiao Tong University
Shanghai, 200240, P. R. China
E-mail address: radjesvarane.alexandre@ecole-navale.fr

Y. Morimoto, Graduate School of Human and Environmental Studies, Kyoto University
Kyoto, 606-8501, Japan
E-mail address: morimoto@math.h.kyoto-u.ac.jp

S. Ukai, 17-26 Iwasaki-cho, Hodogaya-ku, Yokohama 240-0015, Japan
E-mail address: ukai@kurims.kyoto-u.ac.jp

C.-J. Xu, School of Mathematics, Wuhan University 430072, Wuhan, P. R. China
and
Université de Rouen, UMR 6085-CNRS, Mathématiques
Avenue de l'Université, BP.12, 76801 Saint Etienne du Rouvray, France
E-mail address: Chao-Jiang.Xu@univ-rouen.fr

T. Yang, Department of mathematics, City University of Hong Kong, Hong Kong, P. R. China
and
School of Mathematics, Wuhan University 430072, Wuhan, P. R. China
E-mail address: matyang@cityu.edu.hk