ON THE WATER WAVES EQUATIONS WITH SURFACE TENSION

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Conference dedicated to the sixtieth birthday of Professor Yoshinori Morimoto.

1. The purpose of the model

The purpose is to study the dynamics of a fluid

- in a space time region $\Omega \subset \mathbf{R}_t \times \mathbf{R}^{d+1}_{(x,y)}$ with a free boundary Σ
- satisfying the Euler's equations of motion in this domain
- being incompressible and irrotational
- moving under the force of gravitation and having a surface tension.

If $u : \Omega \to \mathbf{R}^{d+1}$ is the velocity field of the fluid then the position $X(t) = (x(t), y(t))$ of a particule at time t starting from a position X_0 at $t = 0$ satisfies the system of differential equations

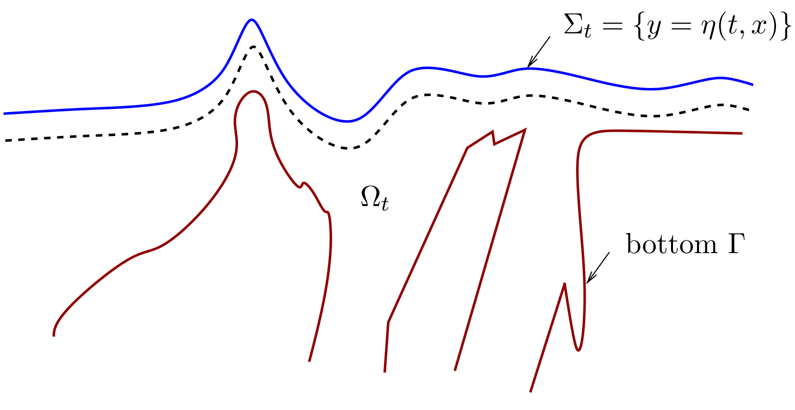
$$\dot{X}(t) = u(t, X(t)), \quad X(0) = X_0.$$

The Euler equations are the traduction of the Newton law $F = m\gamma$. Since $\gamma(t) = \ddot{X}(t) = [\partial_t u + (u \cdot \nabla_X)u](t, X(t))$, u will satisfy the system

$$\begin{cases} \partial_t u + (u \cdot \nabla_X)u + ge_{d+1} + \nabla p = 0, \\ \operatorname{div} u = 0, (\textit{incompressibility}) \\ \operatorname{curl} u = 0, (\textit{irrotationality}) \end{cases}$$

where $g > 0$ is the acceleration of the gravity, e_{d+1} is the last vector of the canonical basis of $\mathbf{R}^{d+1}$ and p the pressure.

The picture is as follows.



Here the surface $\Sigma_t = \{X = (x, y) \in \mathbf{R}^d \times \mathbf{R} : y = \eta(t, x)\}$ is an unknown.

Since div $u = $ curl $u = 0$ we have $u = \nabla_X \phi$, where $\Delta_X \phi = 0$ in the domain. Then ϕ is a solution of

$$\partial_t \phi + \frac{1}{2}|\nabla_X \phi|^2 + gy + p - p_0 = 0 \quad \text{in } \Omega.$$

Moreover we assume that

- any particule which at $t = 0$ is on the surface remains on the surface . This reads

$$\partial_t \eta - (\partial_y \phi - \nabla_x \eta \cdot \nabla_x \phi) = 0 \quad \text{on } \Sigma_t,$$

  Indeed we set $\theta(t) = y(t) - \eta(t, x(t))$. We have $\theta(0) = 0$, we write $\dot{\theta}(t) = 0$ for all t and we use the system of differential equations satisfied by $u = (\nabla_x \phi, \partial_y \phi)$.

Let us remark that

$$(\partial_y \phi - \nabla_x \eta \cdot \nabla_x \phi)|_{\Sigma_t} = \sqrt{1 + |\nabla \eta|^2}(\partial_n \phi)|_{\Sigma_t},$$

where ∂_n is the exterior normal derivative to the surface.

Finally it is assumed that

- the jump of p across Σ_t is proportional to the mean curvature of the surface

i.e.

$$[p] = -\kappa H(\eta) = -\kappa \operatorname{div}\left(\frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}}\right)$$

where $\kappa > 0$ is the surface tension.

Following Zaharov we consider

$$\psi(t, x) = \phi|_{\Sigma_t} = \phi(t, x, \eta(t, x))$$

and we introduce the Dirichlet-Neumann operator,

$$G(\eta)\psi = \sqrt{1 + |\nabla \eta|^2}(\partial_n \phi)|_{\Sigma_t}.$$

Then the system satisfied by (η, ψ) is the following:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0, \\ \partial_t \psi + \dfrac{1}{2}|\nabla \psi|^2 - \dfrac{\big(\nabla \psi \cdot \nabla \eta + G(\eta)\psi\big)^2}{2(1 + |\nabla \eta|^2)} + g\eta - \kappa H(\eta) = 0. \end{cases}$$

This is the "water waves system" .

The purpose of our work is

1. to revisite the Cauchy theory,
2. to show that these equations enjoy dispersive properties such as Kato smoothing effect and Strichartz estimates .

2. The Cauchy Theory ($d \geq 1$)

Theorem 2.1. *Let $d \geq 1$, $s > 2 + \frac{d}{2}$ and $(\eta_0, \psi_0) \in H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)$ such that* $\operatorname{dist}(\Sigma_0, \Gamma) > 0$.

Then there exists $T > 0$ such that the Cauchy problem for (1) *has a unique solution $(\eta, \psi) \in C^0\big([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}^d) \times H^s(\mathbf{R}^d)\big)$ such that* $\operatorname{dist}(\Sigma_t, \Gamma) > 0$ *for all $t \in [0, T]$.*

Concerning this problem there are many results starting from the pionneering works of S. Wu [31], [32], [33] and K. Beyer and M. Günther [9]. See Schweiser [25], D. M. Ambrose and N. Masmoudi [7], D. Lannes, T. Iguchi [20], J. Shatah and C. Zeng [26], M. Ming and Z. Zhang [23], F. Rousset and N. Tzvetkov [24].

Remark 2.2. (i) $s > 2 + \frac{d}{2}$ appears to be the natural threshold of regularity (as it controls the Lipschitz norm of the non-linearities). Indeed

$$\begin{aligned}
&\Big(\psi \in H^s(\mathbf{R}^d), \eta \in H^{s+\frac{1}{2}}(\mathbf{R}^d)\Big) \Rightarrow \phi \in H^{s+\frac{1}{2}}(\Omega) \quad (\Omega \subset \mathbf{R}^{d+1})\\
&\Rightarrow u = \nabla\phi \in H^{s-\frac{1}{2}}(\Omega) \Rightarrow \nabla u \in H^{s-\frac{3}{2}}(\Omega) \subset L^\infty(\Omega) \qquad (2.1)\\
&\text{if } s - \frac{3}{2} > \frac{d+1}{2} \text{ that is, } s > 2 + \frac{d}{2}.
\end{aligned}$$

It is consequently the natural assumption to make as long as dispersive effects are not taken into account .

(ii) Notice that no assumption is made on the bottom.

(iii) The $\frac{1}{2}$-difference in the regularity of η and ψ is natural. Indeed in the case of infinite bottom we have $G(0) = |D_x|$. So (when $g = 0$) the linearized system around $(0, 0)$ is

$$\partial_t\eta - |D_x|\,\psi = 0, \qquad \partial_t\psi - \Delta\eta = 0.$$

If we set, $\Theta = |D_x|^{\frac{1}{2}}\eta + i\psi$ then,

$$(\partial_t + i|D_x|^{\frac{3}{2}})\Theta = 0.$$

This can also be seen on the Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\int G(\eta)\psi\psi dx + \frac{1}{2}\int \eta^2 dx + \kappa\int \frac{|\nabla\eta|^2}{1 + \sqrt{1 + |\nabla\eta|^2}}dx.$$

3. Dispersive estimates

Theorem 3.1 (Kato smoothing effect)**.** *Let* $d = 1$ *,* $s > 2 + 1/2$ *and* $T > 0$*.*

If $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ *is a solution of* (1) *then*

$$\forall\delta > 0, \quad \langle x\rangle^{-\frac{1}{2}-\delta}(\eta, \psi) \in L^2(0, T; H^{s+\frac{1}{2}+\frac{1}{4}}(\mathbf{R}) \times H^{s+\frac{1}{4}}(\mathbf{R})),$$

Theorem 3.2 (Semiclassical Strichartz)**.** *Let* $d = 1$*,* $s > 2 + 1/2$ *and* $T > 0$*.*

If $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ *is a solution of* (1) *then*

$$(\eta, \psi) \in L^4(0, T; W^{s+\frac{1}{4},\infty}(\mathbf{R}) \times W^{s-\frac{1}{2}+\frac{1}{4},\infty}(\mathbf{R}))$$

The last theorem has been recently proved by Christianson-Hur and Staffilani [15] when $s > 15$.

Theorem 3.3 (Classical Strichartz)**.** *Let* $d = 1$*,* $s > 11/2$ *and* $T > 0$*. If* $(\eta, \psi) \in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbf{R}) \times H^s(\mathbf{R}))$ *is a solution of* (1) *then (if* $s + \frac{3}{8} \notin \mathbf{N}$*)*

$$(\eta, \psi) \in L^4(0, T; W^{s+\frac{3}{8},\infty}(\mathbf{R}) \times W^{s-\frac{1}{2}+\frac{3}{8},\infty}(\mathbf{R})).$$

If $s+\frac{3}{8}\in \mathbf{N}$ one has to replace the Hölder space by a Besov space. This result corresponds to the "end point Strichartz estimates" and one can interpolate it with the trivial information:

$$(\eta,\psi)\in L^2(0,T;H^{s+\frac{1}{2}}(\mathbf{R})\times H^s(\mathbf{R})).$$

4. Sketch of the proofs

4.0.1. *Paralinearization of the equations.* We make a great use of Bony's theory of paradifferential operators (see [11]).

We introduce the symbol class Γ^m_ρ for $m\in\mathbf{R}$ et $\rho\geq 0$,

$$a\in\Gamma^m_\rho \Leftrightarrow \|\partial^\alpha_\xi a(\cdot,\xi)\|_{W^{\rho,\infty}(\mathbf{R}^d)}\leq C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

If $a\in\Gamma^m_\rho$ the paradifferential operator T_a is defined by

$$\widehat{T_a u}(\xi)=\int \chi(\xi-\eta,\eta)\hat{a}(\xi-\eta,\eta)\psi(\eta)\hat{u}(\eta)d\eta,$$

where $\chi\in C^\infty(\mathbf{R}^{2d}),\chi(\theta,\eta)=1$ if $|\theta|\leq\varepsilon_1|\eta|$ and $\chi(\theta,\eta)=0$ if $|\theta|\geq\varepsilon_2|\eta|$ et $\psi\in C^\infty(\mathbf{R}^d),\psi(\eta)=0$ if $|\eta|\leq 1$.

For these operators we have a complete symbolic calculus : composition, adjoint, continuity in the scale of Sobolev spaces etc..

For instance, if $a\in L^\infty(\mathbf{R}^d)$ then for all $s\in\mathbf{R}$,

$$\|T_a u\|_{H^s(\mathbf{R}^d)}\leq C\|a\|_{L^\infty(\mathbf{R}^d)}\|u\|_{H^s(\mathbf{R}^d)}.$$

We have a result paralinearisation of a product. If $\rho>0$ then,

$$u,v\in W^{\rho,\infty}(\mathbf{R}^d)\Rightarrow uv=T_u v+T_v u+R(u,v)\quad R\in W^{2\rho,\infty}(\mathbf{R}^d).$$

Consider a solution of the water waves system,

$$(\eta,\psi)\in C^0\big([0,T];H^{s+\frac{1}{2}}(\mathbf{R}^d)\times H^s(\mathbf{R}^d)\big),s>2+\frac{d}{2}.$$

and let us set

$$\mathfrak{B}=\frac{\partial\phi}{\partial y}|_{\Sigma_t},\quad V=\nabla_x\phi|_{\Sigma_t},\quad U=\psi-T_{\mathfrak{B}}\eta.$$

Then the first step in the proof is the following result.

Proposition 4.1 (Main reduction)**.** *There exist a symbol $p=p(t,x,\xi)$ of order $1/2$, a symbol $\gamma=\gamma(t,x,\xi)$ of order $3/2$ and a function $q=q(t,x)$ such that the scalar complex-valued unknown $\Phi\in C^0([0,T];H^s(\mathbf{R}^d))$ defined by*

$$\Phi=T_p\,\eta+iT_q\,U$$

satisfies the equation

$$\partial_t\Phi+T_V\cdot\nabla\Phi+iT_\gamma\Phi=F$$

where

$$\|F\|_{L^\infty(0,T;H^s)}\leq C\left(\|(\eta,\psi)\|_{L^\infty(0,T;H^{s+\frac{1}{2}}\times H^s)}\right).$$

Here are the main difficulties in the sequel:

(i) The subprincipal symbol $T_V\cdot\nabla$ has order only $\frac{1}{2}$ less than the principal part, in contrast with the usual Schrödinger equation.

(ii) The coefficients are of low regularity and time dependent.

(iii) The equation is pseudo-differential.

4.0.2. *The Kato smoothing effect.* For the Kato smoothing effect we use Doi's method (see [17], [18]) which consists in

1. constructing an escape function adapted to our operator,
2. proving a Gårding inequality.

Here are the precise statements.

Lemma 4.2. *One can find a smooth symbol $a = a(x, \xi)$ homogeneous of order 0 in ξ and bounded in x, such that*

$$\forall \delta > 0 \quad \exists K > 0 : \quad \left\{ c\,|\xi|^{\frac{3}{2}}\,, a \right\}(t, x, \xi) \geq K \frac{|\xi|^{1/2}}{\langle x \rangle^{1+2\delta}}.$$

Lemma 4.3. *Assume that q is a symbol of order $\frac{1}{2}$ such that ,*

$$q(t, x, \xi) \geq K \langle x \rangle^{-1-2\delta} \,|\xi|^{\frac{1}{2}}\,, \quad \delta > 0.$$

Then

$$\langle T_q u, u \rangle_{L^2} \geq C_1 \left\| \langle x \rangle^{-\frac{1}{2}-\delta} u \right\|^2_{H^{\frac{1}{4}}} - C_2 \left\| u \right\|^2_{L^2}.$$

Using these Lemmas (when $s = 0$) we conclude the proof of the smoothing effect in computing the quantity $\frac{d}{dt} \langle T_q u, u \rangle_{L^2}$.

4.0.3. *The Strichartz estimates.* To prove the Strichartz estimates we use a classical method:

(i) Littlewood-Paley decomposition in frequency,

(ii) dispersive ($L^1 - L^\infty$) estimates (on each dyadic bloc) by constructing a parametrix,

(iii) the TT^* argument.

The crucial point is of course (ii).

We write $\varphi = \sum_{j=-1}^{+\infty} \varphi_j$ where $\varphi_j = \mathcal{F}^{-1}[\chi(2^{-j}\xi)\hat{\varphi}(\xi)]$, $j \geq 0$, with $\mathrm{supp}\chi \subset \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$.

We set $h = 2^{-j}$ and $v(\sigma, x) = (\Delta_j \varphi)(h^{\frac{1}{2}}\sigma, x)$. Then

$$\begin{cases} h\partial_\sigma v + h^{\frac{1}{2}} T^\delta_W (h\partial_x) v + i\,|hD_x|^{\frac{3}{2}}\, v = F_2, \\ v|_{\sigma=0} = v_0. \end{cases}$$

We look for a parametrix of the form

$$Kv_0(\sigma, x) = \frac{1}{2\pi h} \iint e^{\frac{i}{h}\left((x-y)\eta + \sigma a(\eta) + h^{\frac{1}{2}} \psi(\sigma, x, \eta)\right)} b(\sigma, x, \eta, h) v_0(y)\, d\eta dy,$$

with $a(\eta) = |\eta|^{\frac{3}{2}}$, supp b $\subset \{\eta : \frac{1}{2} \leq |\eta| \leq \frac{3}{2}\}$ and we prove,

$$\| K v_0(\sigma, \cdot) \|_{L^\infty} \leq \underbrace{\frac{C}{h}}_{Sobolev} \underbrace{\left(\frac{h}{\sigma}\right)^{\frac{1}{2}}}_{Dispersive} \| v_0 \|_{L^1}.$$

There are several differences between the semiclassical and classical Strichartz estimates. Indeed,

- in the semiclassical case : $0 < |\sigma| \leq C$ (classical time $|t| \leq Ch^{\frac{1}{2}}$) the phase ψ is the solution of the linear problem

$$\partial_\sigma \psi + a'(\eta) \partial_x \psi = -\eta W^h_\delta, \quad \psi|_{\sigma=0} = 0,$$

$b = \sum h^{j(\frac{1}{2}-\delta)} b_j$ where the $b_j's$ satisfy transport equations and to conclude we use a precise form of the Van der Corput Lemma. Then the Strichartz estimate on a time intervall of size Ch follows and we add all these estimates (which gives rise to a loss of regularity) to obtain the "semiclassical" estimate.

- in the classical case : $0 < |\sigma| \leq Ch^{-\frac{1}{2}}$ (classical time $|t| \leq C$) the phase is the solution of the non linear problem

$$\partial_\sigma \psi - \frac{a(\eta + h^{\frac{1}{2}} \partial_x \psi) - a(\eta)}{h^{\frac{1}{2}}} + h^{\frac{1}{2}} W_h^\delta \partial_x \psi = -\eta W_h^\delta, \quad \psi|_{\sigma=0} = 0,$$

$b = e^\theta$ with $\theta = \sum h^{j\mu_0} \theta_j, \mu_0 > 0$, and we use a stationnary phase method with the complex phase $x\eta + \sigma a'(\eta) + h^{\frac{1}{2}} \psi + \frac{i}{h}\theta$.

5. Final Remarks

1. A direct consequence of this analysis is to lower the regularity threshold for well posedness below $s = 2 + 1/2$, allowing non Lipschitz initial velocities.

2. For the 3-d water-waves:

(i) Smoothing effect applies for rotation invariant initial states.

(ii) In general non trapping assumption is required. i.e. the smoothing effect should be proved false if trapping occurs.

(iii) Semiclassical Strichartz should be true for 3-d. pb: minimal smoothness of initial data? Notice that in 1-d an important simplification is due to the change of variables which reduces matters to $|D_x|^{3/2}$. In general, no such change of variables is possible.

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