Refined diagonalization procedure for C^m coefficients and its applications to the Cauchy problem of hyperbolic equations

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1. Introduction

Energy estimate for the wave equations with variable propagation speed

Consider the following Cauchy problem:

$$(1)\cdots \quad \begin{cases} (\partial_t^2 - a^2\Delta)\, u(t,x) = 0, & (t,x) \in [T,\infty) \times \mathbb{R}^n, \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases}$$

$a(>0)$: propagation speed

$$E(t) := \frac{1}{2}\left(a^2 \, \|\nabla u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2 \right) \quad \text{(total energy, } \|\cdot\| = \|\cdot\|_{L^2}\text{)}$$

$$E'(t) = 0 \iff E(t) \equiv E(T): \text{ energy conservation}$$

Variable propagation speed models

$$\begin{cases} (\partial_t^2 - a(t)^2 \Delta)\, u(t,x) = 0 \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)) \end{cases}$$

$$\begin{cases} (\partial_t^2 + A(t,x,D_x))\, u(t,x) = 0 \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)) \end{cases}$$

$$A(t,x,D_x) := \sum_{1 \le j,k \le n} D_{x_j}\, a_{jk}(t,x)\, D_{x_k}, \quad D_x = -i\partial_x$$

$$E(t) := \left\{ (A(t,\cdot,D)\, u(t,\cdot), u(t,\cdot)) + \|u_t(t,\cdot)\|^2 \right\}, \quad ((\cdot,\cdot) = (\cdot,\cdot)_{L^2})$$

Time dependent propagation speed: $(A(t,x,D_x) = -a(t)^2\Delta)$

Assume $a(t) \in C^1([0,\infty))$, $0 < a_0 \le a(t) \le a_1 < \infty$

$$E'(t) = a'(t)\,a(t)\,\|\nabla u(t,\cdot)\|^2 \begin{cases} \le \frac{2|a'(t)|}{a(t)} E(t) \\ \ge -\frac{2|a'(t)|}{a(t)} E(t) \end{cases}$$

$$\Longrightarrow \quad E(t) \begin{cases} \le \exp\left(\int_0^t \frac{2|a'(\tau)|}{a(\tau)}\,d\tau\right) E(0) \\ \ge \exp\left(-\int_0^t \frac{2|a'(\tau)|}{a(\tau)}\,d\tau\right) E(0) \qquad (a(t) > 0) \end{cases}$$

$$\Longrightarrow \quad {}^{\exists}C_0 E(0) \le E(t) \le {}^{\exists}C_1 E(0) \text{ if } a'(t) \in L^1((0,\infty)).$$

(Generalized energy conservation law = **GECL**)

Theorem. $a'(t) \in L^1((0,\infty)) \Rightarrow$ GECL.

Question. Doesn't GECL hold in general if $a'(t) \notin L^1((0,\infty))$?

Farther problems

$a(t) \notin C^1([0,\infty))$

- The Cauchy problem is not locally L^2 well-posed in general;
- If $a(t)$ is Hölder continuous, then the Cauchy problem is Gevrey well-posed;

$a(t) \not> 0$

- The Cauchy problem is not locally L^2 well-posed in general even if $a(t) \in C^1([0,\infty))$;
- If $a(t) \in C^{m,\alpha}([0,\infty))$, then the Cauchy problem is Gevrey well-posed with a corresponding order;

Cf. [CDS] (strictly hyperbolic**)**, [CJS] (weakly hyperbolic**)**

Overview of this talk

We introduce a new argument for the analysis of second order hyperbolic equation with time dependent coefficients, which will be called the C^m property. Actually, we shall consider the following equations:

$$(\partial_t^2 - a(t)^2\Delta)\, u = 0$$ ・・・・・・ (Wave equation with variable propagation speed)

$$(\partial_t^2 - \Delta + b(t)\,\partial_t)\, u = 0$$ ・・・・・・ (Dissipative wave equation)

$$(\partial_t^2 - a(t)^2\Delta + \textstyle\sum_{j=1}^n b_j(t)\,\partial_{x_j})\, u = 0$$

$$(\partial_t^2 + A_p(t, D_x))\, u = 0$$ ・・・・・・ (p-evolution type equation)

and discuss about the asymptotic stabilization of the solutions: GECL, well-posedness, decay property, and etc.

Section 2: GECL on C^2 and C^m properties with some stabilization property of the coefficients to the equation:

$$(\partial_t^2 - a(t)^2\Delta)\, u = 0 \qquad \cdots (1)$$

***Keywords**: C^2 property, C^m property, stabilization property, pseudo-differential zone "$Z_{\boldsymbol{\psi}}$", hyperbolic zone "Z_H", WKB solutions.*

Section 3, 4: Gevrey, C^{∞} and L^2 well-posedness for (1) with non-Lipschitz continuous coefficients (in Section 4) and degenerate coefficient at one point.

***Keywords**: stabilization property for degenerate coefficient, intermediate zone.*

Section 5: GECL on C2 and Cm property with increasing propagation speed.

Section 6: Open problems

2. GECL for the wave equations with C^m coefficients

Consider the following Cauchy problem and corresponding energy:

$$\textbf{(1)} \cdots \begin{cases} (\partial_t^2 - a(t)^2 \Delta)\, u(t,x) = 0 \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)) \end{cases}$$

$$E(t) := \frac{1}{2}\left(a(t)^2 \|\nabla u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2 \right)$$

under the assumptions:

$$a(t) \in C^m([0,\infty)) \ \ (m \geq 2), \quad 0 < a_0 \leq a(t) \leq a_1 < \infty$$

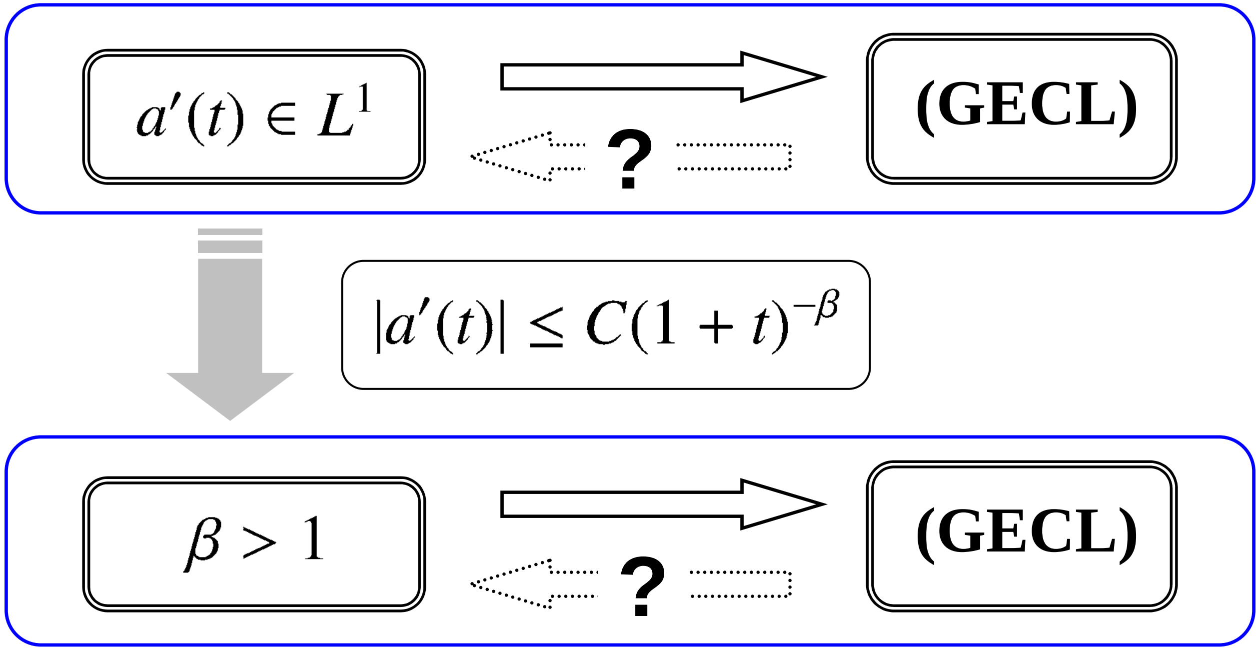
we have observed

$$E'(t) \begin{cases} \geq 0 \quad \text{for } \ a'(t) > 0 \quad \Rightarrow \quad (E(t) \nearrow) \\ \leq 0 \quad \text{for } \ a'(t) < 0 \quad \Rightarrow \quad (E(t) \searrow) \end{cases}$$

$\Longrightarrow$ the previous estimates: $-\dfrac{2|a'(t)|}{a(t)}E(t) \leq E'(t) \leq \dfrac{2|a'(t)|}{a(t)}E(t)$ is too rough!

Can we make a consideration the sign of $a'(t)$?

Problem establishment



Problem. How is the condition for GECL described by β ?

Brief conclusion. β =1 is the critical for m=1. But β can be taken smaller than 1 if m≧2 under assuming some additional assumptions for higher order derivatives and introducing a new property for the control of amplitude; ***stabilization property***.

C² property (cf. [RS])

(1) Fourier tf. w.r.t. x

$$\begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)\, v(t,\xi) = 0 \\ (v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi)) \end{cases}$$

$$\mathcal{E}(t,\xi) := \frac{1}{2}\left(a(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right)$$

GECL is not easily proved for the critical case β =1. Indeed, we observe that

$$\partial_t \mathcal{E}(t,\xi) = a(t)a'(t)|\xi|^2|v(t,\xi)|^2 \le M(1+t)^{-1}\mathcal{E}(t,\xi)$$

$$\Longrightarrow \quad \mathcal{E}(t,\xi) \le \exp\left(M \int_0^t (1+\tau)^{-1}\, d\tau \right) \mathcal{E}(0,\xi)$$

$$\le (1+t)^M\, \mathcal{E}(0,\xi)$$

("log effect" makes a problem!)

Idea of the C^2 property and log effect

$$a(t) \equiv a \text{ (constant)}$$

$$v(t,\xi) = \frac{a|\xi|v_0 - iv_1}{2a|\xi|}\, e^{ia|\xi|t} + \frac{a|\xi|v_0 + iv_1}{2a|\xi|}\, e^{-ia|\xi|t}$$

It will be reasonable supposing that the first approximation of the solution for variable coefficient is given by

$$v(t,\xi) = \tilde{A}(t,\xi)\left(\frac{a(t)|\xi|v_0 - iv_1}{2a(t)|\xi|}\, e^{i|\xi|\int_0^t a(\tau)\,d\tau} + \frac{a(t)|\xi|v_0 + iv_1}{2a(t)|\xi|}\, e^{-i|\xi|\int_0^t a(\tau)\,d\tau}\right)$$

・・・ (WKB solution)

⇕

$$V_1(t,\xi) = \begin{pmatrix} v_t(t,\xi) + ia(t)|\xi|v(t,\xi) \\ v_t(t,\xi) - ia(t)|\xi|v(t,\xi) \end{pmatrix} = \tilde{\tilde{A}}(t,\xi)V_1(0,\xi)$$

・・・ (system)

⇕

$$\mathcal{E}(t,\xi) = A(t,\xi)\,\mathcal{E}(0,\xi)$$

・・・ (energy)

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0$$

(reduction to first order system)

$$\partial_t V = (B+R)V \qquad V = \begin{pmatrix} ia|\xi|v \\ v_t \end{pmatrix},\ B = ia|\xi|\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ R = \frac{a'}{a}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(diagonalization; under hyperbolicity)

$$\partial_t V_1 = (\Lambda_1 + R_1)V_1 \qquad \Lambda_1 = ia|\xi|\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ R_1 = \frac{a'}{2a}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Introduction of the usual energy is corresponding to diagonalization under the hyperbolicity of the equation with C^1 coefficient.

$$\boxed{\partial_t V_1 = (\Phi_1 + B_1)V_1}$$

$$\Phi_1 = \begin{pmatrix} ia|\xi| + \frac{a'}{2a} & 0 \\ 0 & -ia|\xi| + \frac{a'}{2a} \end{pmatrix} =: \begin{pmatrix} \phi_+ & 0 \\ 0 & \phi_- \end{pmatrix}$$

$$B_1 = \frac{a'}{2a}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} =: \begin{pmatrix} 0 & \beta_1 \\ \beta_1 & 0 \end{pmatrix}$$

The order $|a'(t)| \leq C(1+t)^{-1}$ of the anti-diagonal entries cause log effect.

Reduce the equation of V_1 into the equation of V_2 by some diagonalization:

$$\boxed{\partial_t V_2 = (\Phi_1 + R_2)V_2}$$

here R_2 must satisfy the property:

R_2: the integral in L^1 does not bring **log effect**.

$$\partial_t V_1 = (\Phi_1 + B_1)V_1 \quad \Longrightarrow \quad \partial_t V_2 = (\Phi_1 + R_2)V_2$$

$$M_1 := \begin{pmatrix} 1 & p_+ \\ p_- & 1 \end{pmatrix}$$

$$V_2 := M_1^{-1}V_1, \ \Phi_1 + R_2 = M_1^{-1}(\Phi_1 + B_1)M_1 - M_1^{-1}[\partial_t, M_1]$$

$$R_2 = \frac{1}{1-p_+p_-}\begin{pmatrix} p_+p_-(\phi_+ - \phi_-) - \beta_1(p_+ - p_-) & p_+(\phi_+ - \phi_-) + \beta_1(1 - p_+^2) \\ -p_-(\phi_+ - \phi_-) + \beta_1(1 - p_-^2) & -p_+p_-(\phi_+ - \phi_-) + \beta_1(p_+ - p_-) \end{pmatrix}$$

$$+ \frac{1}{1-p_+p_-}\begin{pmatrix} -p'_-p_+ & p'_+ \\ p'_- & -p'_+p_- \end{pmatrix}$$

$p_\pm$ must be chosen providing the following properties:

$^\exists M_1^{-1}$: $\quad |p_+p_-| \le \frac{1}{2}$;

$p_+p_-(\phi_+-\phi_-)-\beta_1(p_+-p_-)$, $\quad \pm p_\pm(\phi_+-\phi_-)+\beta_1(1-p_\pm^2)$, $\quad p'_\pm$, $\quad p'_\pm p_\mp$:

lower order such as log effect does not arise;

$$p_{\pm} := \pm \frac{-i\,\beta_1}{i\,(\phi_+ - \phi_-)} = \mp \frac{i\,\beta_1}{2a|\xi|} = \pm \frac{i\,a'}{4a^2|\xi|} =: \pm\, i\, q, \qquad |q| \le \frac{C}{(1+t)|\xi|}$$

$$\det M_1 = 1 - q^2 \ge 1 - \left(\frac{C}{(1+t)|\xi|}\right)^2$$

$$p_+ p_-(\phi_+ - \phi_-) - \beta_1(p_+ - p_-) = -2\, i\, a\, q^2|\xi|$$

$$\pm p_{\pm}(\phi_+ - \phi_-) + \beta_1(1 - p_{\pm}^2) = 2\, a\, q^3\, |\xi|$$

$$|R_2| \le C|\xi|^{-1}\left((1+t)^{-2} + |a''|\right)$$

$$|a''(t)| \le C(1+t)^{-2}$$

$$|R_2| \le C|\xi|^{-1}(1+t)^{-2}$$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1$$

$$\partial_t V_2 = (\Phi_1 + R_2)V_2$$

$$W_2 := \Theta_1^{-1} V_2 := \sqrt{\frac{a(t_0)}{a(t)}}\,\Theta^{-1} V_2, \quad \tilde{R}_2 = \Theta^{-1} R_2 \Theta$$

$$\Theta_1 := \begin{pmatrix} \exp\left(\int_{t_0}^{t} \phi_+(\tau,\xi)\,d\tau\right) & 0 \\ 0 & \exp\left(\int_{t_0}^{t} \phi_-(\tau,\xi)\,d\tau\right) \end{pmatrix}$$

$$\Theta := \begin{pmatrix} e^{i|\xi| \int_{t_0}^{t} a(\tau)\,d\tau} & 0 \\ 0 & e^{-i|\xi| \int_{t_0}^{t} a(\tau)\,d\tau} \end{pmatrix}$$

$$\partial_t W_2 = \tilde{R}_2 W_2$$

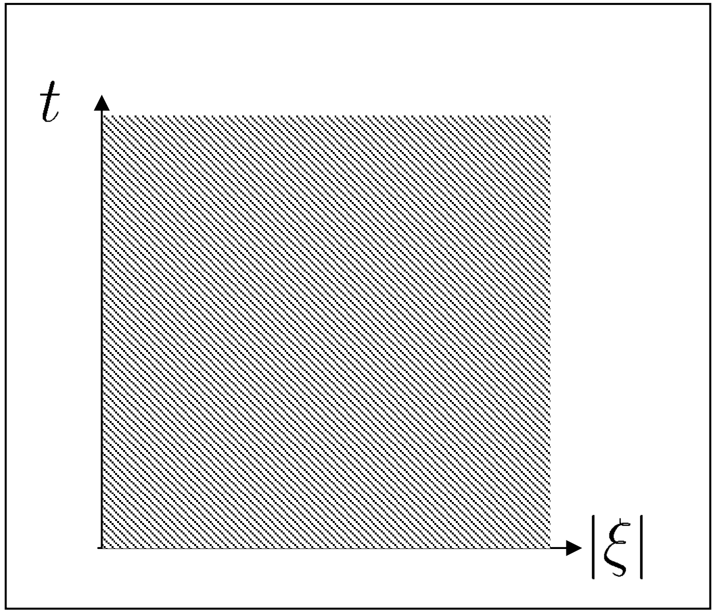
elliptic t.f.

$$|W_2(t,\xi)| \le \exp\left(C \int_{t_0}^{t} |R_2(\tau,\xi)|\,d\tau\right) |W_2(t_0,\xi)|$$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1 \quad \Longrightarrow \quad \partial_t W_2 = \tilde{R}_2 W_2$$

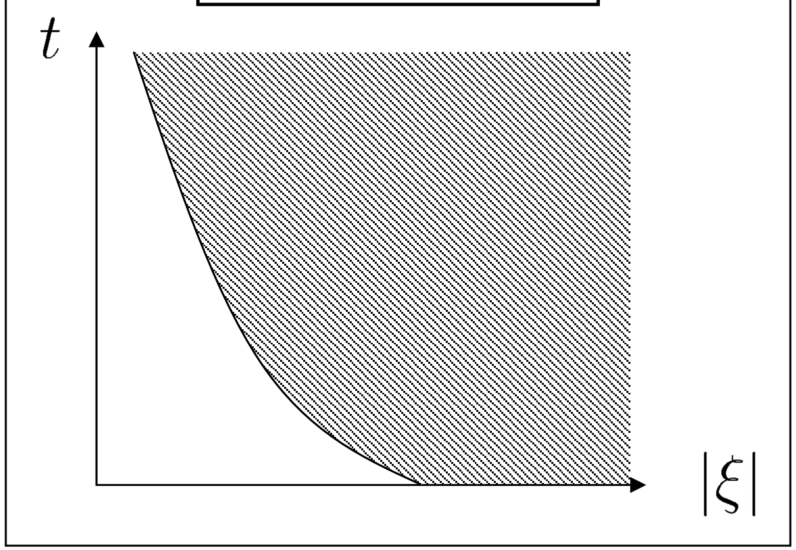
$$(1+t)|\xi| \geq K \;\; (K \gg 1) \Longrightarrow {}^{\exists}M_1 \Longrightarrow |W_2|^2 \simeq \mathcal{E}(t,\xi)$$

$$|a'(t)| \leq \frac{C}{1+t}, \; |a''(t)| \leq \frac{C}{(1+t)^2} \Longrightarrow |\tilde{R}_2| = |R_2| \leq C|\xi|^{-1}(1+t)^{-2}$$



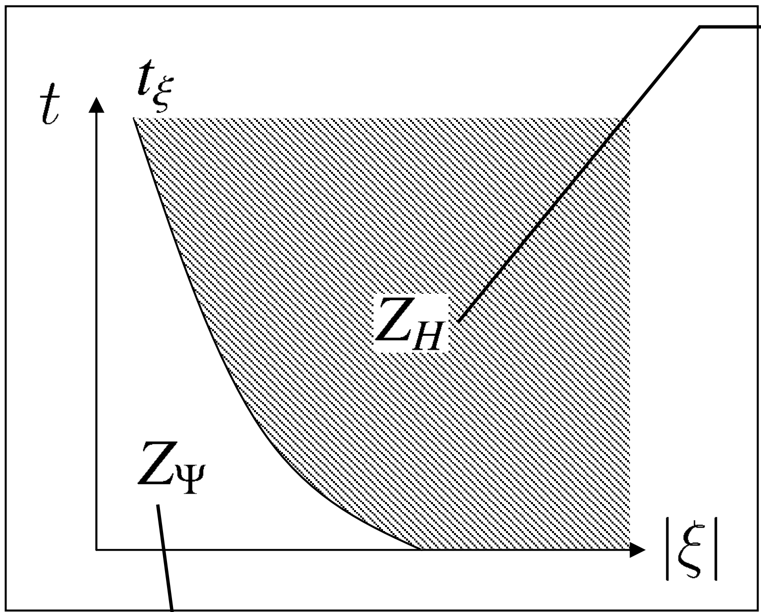
$$|R_1| \leq C(1+t)^{-1}$$

$$(1+t)|\xi| \geq K$$



$$|R_2| \leq C|\xi|^{-1}(1+t)^{-2}$$

$$\begin{cases} Z_H := \left\{(t,\xi)\ ;\ t \geq t_\xi\right\} & \text{(hyperbolic zone)} \\ Z_\Psi := \left\{(t,\xi)\ ;\ t < t_\xi\right\} & \text{(pseudo-differential zone)} \\ (1 + t_\xi)|\xi| = K & \end{cases}$$



$$|R_2| \leq C|\xi|^{-1}(1+t)^{-2}$$

⇓

$$\int_{t_\xi}^{t} |R_2|\, d\tau \leq C|\xi|^{-1}(1+t_\xi)^{-1} = CK^{-1}$$

⇓

$$\mathcal{E}(t,\xi) \leq C\mathcal{E}(t_\xi,\xi) \quad (t \geq t_\xi)$$

??

Estimate in Z_Ψ

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0$$

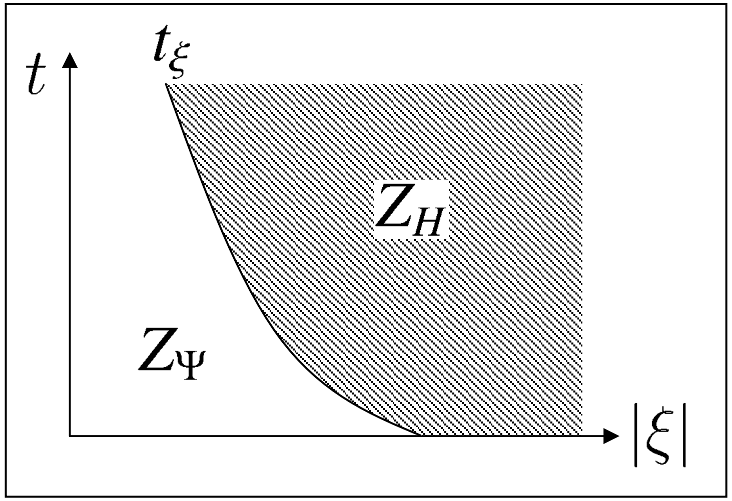
$$\partial_t W_0 = \tilde{R}_0 W_0$$

$$W_0 = \Theta_0 V_0,\ \ V_0 = \begin{pmatrix} v_t(t,\xi) + ia_0|\xi|v(t,\xi) \\ v_t(t,\xi) - ia_0|\xi|v(t,\xi) \end{pmatrix},\ \ \Theta_0 = \begin{pmatrix} e^{ia_0 t|\xi|} & 0 \\ 0 & e^{-ia_0 t|\xi|} \end{pmatrix},$$

$$R_0 = \frac{i\left(a(t)^2 - a_0^2\right)|\xi|}{2a_0}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\ \ \tilde{R}_0 = \Theta_0^{-1} R_0 \Theta_0$$

$$|W_0(t,\xi)| \le \exp\left(C|\xi| \int_0^t |a(\tau) - a_0|\, d\tau\right)|W_0(0,\xi)| \le e^{C|\xi|(1+t)}|W_0(0,\xi)|$$

$$\mathcal{E}(t,\xi) \le C\mathcal{E}(0,\xi) \quad (t \le t_\xi)$$



$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0 \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)) \end{cases}$$

Theorem ([RS])

$a(t) \in C^2([0,\infty))$ $(m \geq 2)$, $0 < a_0 \leq a(t) \leq a_1 < \infty$,

$|a'(t)| \leq C(1+t)^{-1}$, $|a''(t)| \leq C(1+t)^{-2}$

⟹ (GECL)

Remark. (Estimates from below)

$$\begin{cases} |W_2(t,\xi)| \geq \exp\left(-C \displaystyle\int_{t_0}^{t} |R_2(\tau,\xi)|\, d\tau\right) |W_2(t_0,\xi)| \quad \text{in } Z_H; \\ |V_0(t,\xi)| \geq \exp\left(-C|\xi|(1+t)\right) |V_0(0,\xi)| \quad \text{in } Z_\Psi; \end{cases}$$

⟹ $\mathcal{E}(t,\xi) \geq C\mathcal{E}(0,\xi) \quad (t \geq 0)$

Summarization of the proof

Introduce the symbol classes in Z_H:

$$S^p_q := \left\{ f(t,\xi)\ ;\ \left|\partial_t^k f(t,\xi)\right| \le C_k |\xi|^p (1+t)^{-q},\ t \ge t_\xi \right\}$$

$$S^p_q \subset S^{p+\delta}_{q-\delta} \quad (\delta > 0) \quad \text{(hierarchy of the symbol class)}$$

In Z_H

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0$$

reduction to first order system

$$\partial_t V = (B+R)V \qquad B + R \in S^1_0$$

diagonalization with hyperbolicity and C^1 prop.

$$\partial_t V_1 = (\Phi_1 + B_1)V_1 \qquad B_1 \in S^0_1$$

diagonalization by C^2 prop.

$$\partial_t V_2 = (\Phi_1 + R_2)V_2 \qquad R_2 \in S^{-1}_2$$

elliptic t.f.

$$\partial_t W_2 = \tilde{R}_2 W_2 \qquad \tilde{R}_2 \in S^{-1}_2,\ |W_2|^2 \simeq \mathcal{E}$$

In ZΨ

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0$$

reduction to first order system

$$\partial_t W_0 = \tilde{R}_0 W_0$$

diagonalization with hyperbolicity

elliptic t.f.

$$|W_0(t,\xi)|^2 \simeq \mathcal{E}(t,\xi)$$

$$\int_0^{t_\xi} |\tilde{R}_0|\, d\tau \le C|\xi| \int_0^{t_\xi} |a(\tau) - a_0|\, d\tau \le C|\xi|(1 + t_\xi) \le CK^{-1}$$

Remark. Actually, the diagonalization taking into account the hyperbolicity is not necessary for the estimate in ZΨ, but this argument will be useful.

C^m property and GECL

Motivation. C^2 property contributed to prove GECL in the critical case. Does C^m property for m≧3 give some contributions for GECL?

Conjecture. $^\exists\beta_m < 1\ (m \geq 3),\ \ \beta_m \searrow$ as $m \nearrow$, *s.t.*

$$\left|a^{(k)}(t)\right| \leq C_k(1+t)^{-k\beta_m}\ (k = 1, \cdots, m) \implies \text{(GECL)}$$

Such a property does not hold in general (counter examples exist)!

Examples 1 (cf. [RS])

$$a(t) = 2 + \cos\left((\log(1+t))^\gamma\right)$$

$\implies$ (GECL) does not hold for any $\gamma > 1$.

Theorem (cf. [HR1])

$$a(t) \in C^{1,\varepsilon}([0,\infty))\ (\varepsilon > 0),\ \ |a'(t)| \le C(1+t)^{-1},$$

$$\sup_{0<h\ll 1} \left\{ \frac{|a'(t+h) - a'(t)|}{h^{\varepsilon}} \right\} \le C(1+t)^{-1-\varepsilon} \Longrightarrow \textbf{(GECL)}$$

Not C^2 but $C^{1,\varepsilon}$ property is essential!

Indeed, the previous conjecture is not true as itself, but it becomes true under the additional assumption, which will be called the *stabilization property*.

Refined diagonalization procedure with C^m property [H3]

$$\partial_t V_1 = (\Phi_1 + B_1)V_1 \quad \Longrightarrow \quad \partial_t V_2 = (\Phi_2 + B_2)V_2$$

$$\Phi_1 = \begin{pmatrix} \phi_{1+} & 0 \\ 0 & \phi_{1-} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \overline{\beta_1} \\ \beta_1 & 0 \end{pmatrix}, \quad \beta_1 = \overline{\beta_1} = -\frac{a'}{2a}, \quad \phi_{1\pm} = \pm ia|\xi| + \frac{a'}{2a}$$

$$M_1 = \begin{pmatrix} 1 & \frac{-i\beta_1}{d_1} \\ -\frac{-i\beta_1}{d_1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \overline{\left\{\frac{i\beta_1}{d_1}\right\}} \\ \frac{i\beta_1}{d_1} & 1 \end{pmatrix}, \quad d_1 = i(\phi_{1+} - \phi_{1-}) \ \text{(real valued)}$$

$${}^{\exists}M_1$$

$$\left|\frac{\beta_1}{\phi_{1+} - \phi_{1-}}\right| \ll 1$$

$$\Phi_2 = \begin{pmatrix} \phi_{2+} & 0 \\ 0 & \phi_{2-} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \beta_{2-} \\ \beta_{2+} & 0 \end{pmatrix}$$

$$M_1^{-1}\Phi_1 M_1 = \Phi_1 - B_1 + \frac{1}{\det M_1}\begin{pmatrix} -i\frac{|\beta_1|^2}{d_1} & \overline{\frac{-\beta_1|\beta_1|^2}{d_1^2}} \\ \frac{-\beta_1|\beta_1|^2}{d_1} & i\frac{|\beta_1|^2}{d_1} \end{pmatrix}$$

$$M_1^{-1}B_1 M_1 = B_1 + \frac{1}{\det M_1}\begin{pmatrix} \overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\} - \beta\overline{\left\{\frac{i\beta_1}{d_1}\right\}} & -\beta_1\overline{\left\{\frac{i\beta_1}{d_1}\right\}}^2 + \overline{\beta_1}\left|\frac{i\beta_1}{d_1}\right|^2 \\ -\overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\}^2 + \beta_1\left|\frac{i\beta_1}{d_1}\right|^2 & -\left(\overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\} - \beta\overline{\left\{\frac{i\beta_1}{d_1}\right\}}\right) \end{pmatrix}$$

$$= B_1 + \frac{1}{\det M_1}\begin{pmatrix} 2i\Im\left\{\overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\}\right\} & -2i\overline{\left\{\frac{i\beta_1}{d_1}\right\}}\Im\left\{\beta_1\overline{\left\{\frac{i\beta_1}{d_1}\right\}}\right\} \\ -2i\left\{\frac{i\beta_1}{d_1}\right\}\Im\left\{\overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\}\right\} & -2i\Im\left\{\overline{\beta_1}\left\{\frac{i\beta_1}{d_1}\right\}\right\} \end{pmatrix}$$

$$= B_1 + \frac{1}{\det M_1}\begin{pmatrix} 2i\frac{|\beta_1|^2}{d_1} & \overline{\frac{2\beta_1|\beta_1|^2}{d_1^2}} \\ \frac{2\beta_1|\beta_1|^2}{d_1^2} & -2i\frac{|\beta_1|^2}{d_1} \end{pmatrix}$$

$$\boxed{d_1 = i(\phi_{1+} - \phi_{1-})}$$

$$M_1^{-1}\partial_t M_1 = \frac{1}{\det M_1}\begin{pmatrix} -\left\{\frac{i\beta_1}{d_1}\right\}'\overline{\frac{i\beta_1}{d_1}} & \left\{\overline{\frac{i\beta_1}{d_1}}\right\}' \\ \left\{\frac{i\beta_1}{d_1}\right\}' & -\left\{\overline{\frac{i\beta_1}{d_1}}\right\}'\frac{i\beta_1}{d_1} \end{pmatrix}$$

$$= \frac{1}{\det M_1}\begin{pmatrix} -\frac{1}{2}\left(\left|\frac{\beta_1}{d_1}\right|^2\right)' + i\Im\left\{\frac{\beta_1}{d_1}\left\{\overline{\frac{\beta_1}{d_1}}\right\}'\right\} & \left\{\overline{\frac{i\beta_1}{d_1}}\right\}' \\ \left\{\frac{i\beta_1}{d_1}\right\}' & -\frac{1}{2}\left(\left|\frac{\beta_1}{d_1}\right|^2\right)' - i\Im\left\{\frac{\beta_1}{d_1}\left\{\overline{\frac{\beta_1}{d_1}}\right\}'\right\} \end{pmatrix}$$

$$\phi_{2\pm} := \phi_{1\pm} + \frac{\left(\left|\frac{\beta_1}{d_1}\right|^2\right)'}{2\left(1-\left|\frac{\beta_1}{d_1}\right|^2\right)} \pm \frac{i}{\det M_1}\left(\frac{|\beta_1|^2}{d_1} - \Im\left\{\frac{\beta_1}{d_1}\left\{\overline{\frac{\beta_1}{d_1}}\right\}'\right\}\right)$$

$$\beta_2 := \frac{1}{\det M_1}\left\{\frac{\beta_1|\beta_1|^2}{d_1^2} + \left\{\frac{i\beta_1}{d_1}\right\}'\right\} = \beta_{2+} = \overline{\beta_{2-}}$$

Summarization of the proof

$$\boxed{\partial_t V_2 = (\Phi_2 + B_2)V_2} \quad \Phi_2 = \begin{pmatrix} \phi_{2+} & 0 \\ 0 & \phi_{2-} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \overline{\beta_2} \\ \beta_2 & 0 \end{pmatrix}$$

$$\left|\frac{\beta_1}{d_1}\right| \ll 1, \quad d_1 = i(\phi_{1+} - \phi_{1-}) \ \ \text{(real valued)}$$

$$\begin{cases} \phi_{2\pm} = \phi_{1\pm} - \dfrac{1}{2}\left(\log\left(1 - \left|\dfrac{\beta_1}{d_1}\right|^2\right)\right)' \pm \dfrac{i}{\det M_1}\left(\dfrac{|\beta_1|^2}{d_1} - \Im\left\{\dfrac{\beta_1}{d_1}\left(\overline{\dfrac{\beta_1}{d_1}}\right)'\right\}\right) \\ \beta_2 = \dfrac{1}{\det M_1}\left\{\dfrac{\beta_1|\beta_1|^2}{d_1^2} + \left\{\dfrac{i\beta_1}{d_1}\right\}'\right\} \end{cases}$$

$d_2 := i(\phi_{2+} - \phi_{2-})$: real valued

$\Re\{\phi_{2\pm} - \phi_{1\pm}\}$: Riemann integrable

$\beta_2 \in S_2^{-1}$

General case with C^m property

$$\left|a^{(k)}(t)\right| \le C_k(1+t)^{-\beta} \quad (k=1,\cdots,m), \quad (1+t_\xi)^\beta|\xi| = K \gg 1$$

$$S^p_q := \left\{ f(t,\xi)\ ;\ \left|\partial_t^k f(t,\xi)\right| \le C_k|\xi|^p(1+t)^{-\beta q},\ t \ge t_\xi \right\}$$

$$S^p_q \subset S^{p+\delta}_{q-\delta} \quad (\delta > 0)$$

$$\boxed{\partial_t V_k = (\Phi_k + B_k)V_k} \Rightarrow \boxed{\partial_t V_{k+1} = (\Phi_{k+1} + B_{k+1})V_{k+1}}$$

$$\Phi_k = \begin{pmatrix} \phi_{k+} & 0 \\ 0 & \phi_{k-} \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & \overline{\beta_k} \\ \beta_k & 0 \end{pmatrix}$$

$$\Phi_{k+1} = \begin{pmatrix} \phi_{(k+1)+} & 0 \\ 0 & \phi_{(k+1)-} \end{pmatrix}, \quad B_k = \begin{pmatrix} 0 & \overline{\beta_{k+1}} \\ \beta_{k+1} & 0 \end{pmatrix}$$

$$M_k := \begin{pmatrix} 1 & \overline{\left\{\frac{i\beta_k}{d_k}\right\}} \\ \frac{i\beta_k}{d_k} & 1 \end{pmatrix}, \quad d_k := i(\phi_{k+} - \phi_{k-})$$

$\beta_k \in S_k^{-k+1}, \quad d_k^{-1} \in S_0^{-1}, \quad d_k :$ real valued

$$\Rightarrow \begin{cases} \phi_{(k+1)\pm} = \phi_{k\pm} - \dfrac{1}{2}\left(\log\left(k - \left|\dfrac{\beta_k}{d_k}\right|^2\right)\right)' \pm \dfrac{i}{\det M_k}\left(\dfrac{|\beta_k|^2}{d_k} - \Im\left\{\dfrac{\beta_k}{d_k}\left\{\overline{\dfrac{\beta_k}{d_k}}\right\}'\right\}\right) \\ \beta_{k+1} = \dfrac{1}{\det M_k}\left\{\dfrac{\beta_k|\beta_k|^2}{d_k^2} + \left\{\dfrac{i\beta_k}{d_k}\right\}'\right\} \end{cases}$$

$$\Rightarrow \begin{cases} d_{k+1} := i(\phi_{(k+1)+} - \phi_{(k+1)-}) & \text{: real valued} \\ \Re\left\{\phi_{(k+1)\pm} - \phi_{k\pm}\right\} & \text{: Riemann integrable} \\ \beta_{k+1} \in S_{k+1}^{-k} \end{cases}$$

$$\phi_{(k+1)\pm} = \boxed{\phi_{1\pm}} + \boxed{\sum_{j=1}^{k} \Re\left\{\phi_{(j+1)\pm} - \phi_{j\pm}\right\}} + \boxed{\sum_{j=1}^{k} i\Im\left\{\phi_{(j+1)\pm} - \phi_{j\pm}\right\}}$$

$\phi_{1\pm}$: $\pm ia|\xi| + \dfrac{a'}{2a}$

Riemann integrable

pure imaginary

$$\Rightarrow \left|\exp\left(\int_{t_\xi}^{t} \phi_{(k+1)\pm}(\tau,\xi)\,d\tau\right)\right| \le C \Rightarrow |\Theta_{k+1}| + |\Theta_{k+1}^{-1}| \le C,$$

$$\Theta_{k+1} := \begin{pmatrix} \exp\left(\int_{t_\xi}^{t} \phi_{(k+1)+}(\tau,\xi)\,d\tau\right) & 0 \\ 0 & \exp\left(\int_{t_\xi}^{t} \phi_{(k+1)-}(\tau,\xi)\,d\tau\right) \end{pmatrix}$$

$$\boxed{\left(\partial_t^2 + a(t)^2|\xi|^2\right) v = 0} \Rightarrow \boxed{\partial_t W_m = \tilde{R}_m W_m} \quad (t \ge t_\xi)$$

$$|\tilde{R}_m| \le C|\xi|^{-(m-1)}(1+t)^{-m\beta} \quad \left(\Leftarrow\ S_m^{-(m-1)}\right)$$

$$|W_m(t,\xi)|^2 \simeq \mathcal{E}(t,\xi)$$

$$|\tilde{R}_m| \le C|\xi|^{-(m-1)}(1+t)^{-m\beta}, \quad |W_m(t,\xi)|^2 \simeq \mathcal{E}(t,\xi)$$

$$\mathcal{E}(t,\xi) \le \exp\left(C\int_{t_\xi}^{t} |\tilde{R}_m(\tau,\xi)|\,d\tau\right)$$

$$\le \exp\left(C|\xi|^{-m+1}(1+t_\xi)^{-m\beta+1}\right)\mathcal{E}(t_\xi,\xi)$$

$$= \exp\left(CK^{-m+1}(1+t_\xi)^{-\beta+1}\right)\mathcal{E}(t_\xi,\xi)$$

$\mathcal{E}(t,\xi) \le C\mathcal{E}(t_\xi,\xi) \iff \beta \ge 1$ (No improvement from C^2 property!)

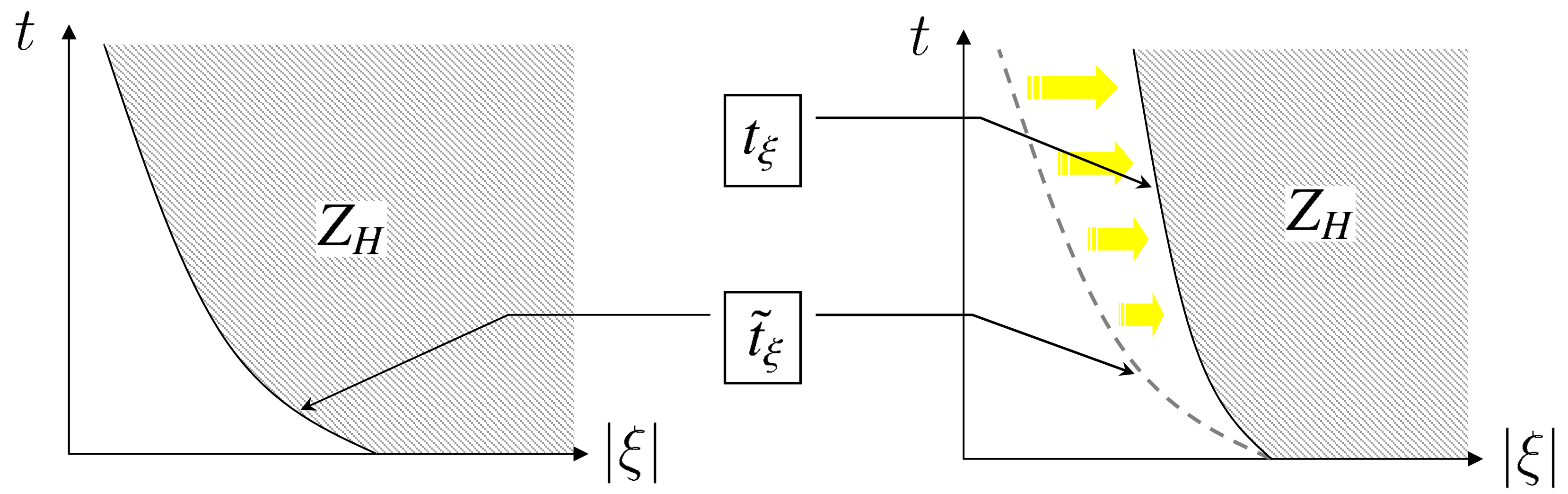
Observation

Generally, it should be expected a better estimate of $\mathcal{E}(t,\xi)$ from the contribution S_m^{-m+1} ($\subset S_2^{-1}$) for $m \ge 3$.

Actually, if $f \in S_m^{-m+1}$ for $m \ge 3$, then we expect a better property of f than the case of $f \in S_2^{-1}$, but it is not true on the border (t_ξ, ξ).

$$(1+\tilde{t}_\xi)^{\beta}|\xi| = K$$

$$(1+t_\xi)^{\kappa\beta}|\xi| = K,\ \ 0<\kappa<1$$



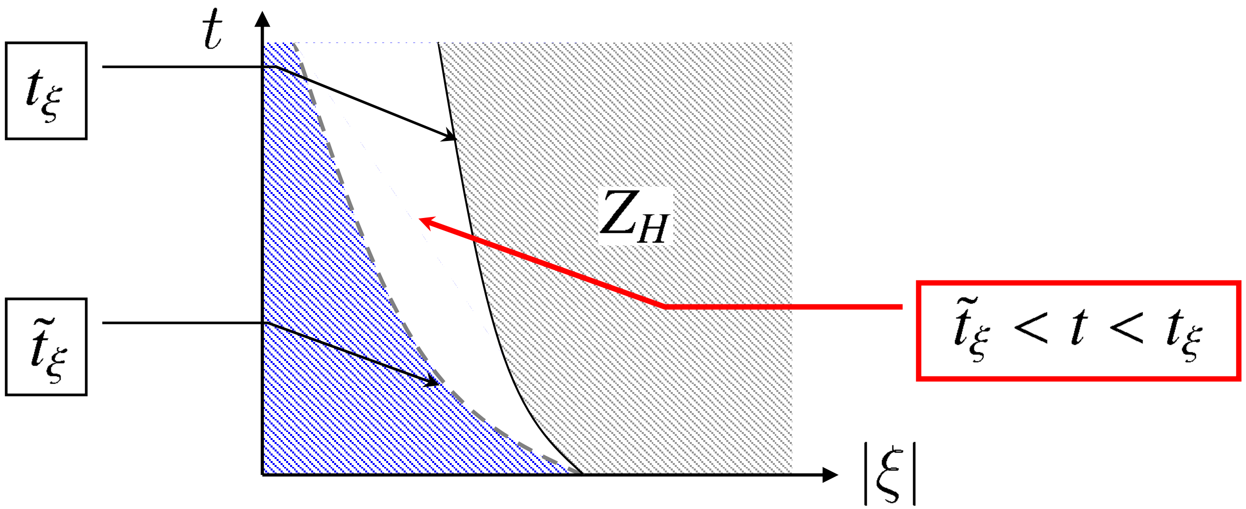
$$|\xi|^{-m+1}(1+t_\xi)^{-m\beta+1} = K^{-m+1}(1+t_\xi)^{\kappa\beta(m-1)-m\beta+1} \le C$$

$$\Longleftarrow\ \beta \ge \frac{1}{m(1-\kappa)+\kappa} =: \beta_{m,\kappa}$$

$\beta_{m,\kappa} < 1$ for $m>1$ and $\kappa<1$.

How do we estimate $\mathcal{E}(t,\xi)$ in the extended Z_Ψ?

$$(1+t_\xi)^{\kappa\beta}|\xi| = K,\ \ 0<\kappa<1$$



We introduce a new property of $a(t)$ for the estimate of $\tilde{t}_\xi < t < t_\xi$; we shall call the property *stabilization property*.

Stabilization property:

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le C(1+t)^\alpha \quad (0 \le \alpha < 1)$$

$$\int_0^t |a(\tau) - a_\infty| \, d\tau \leq C(1+t)^\alpha \quad (0 \leq \alpha < 1)$$

Remark.

- No stabilization property $\Longleftrightarrow$ $\alpha = 1$;
- The constant a_∞ is uniquely determined;
- The counter example by [Reissig - Smith]:

$$a(t) = 2 + \cos\left((\log(1+t))^\gamma\right)$$

  does not satisfy the stabilization property;
- The stabilization property does not require the existence $\lim_{t\to\infty} a(t)$.

Estimate in Z_Ψ under the stabilization property

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0 \quad \Rightarrow \quad \partial_t W_0 = \tilde{R}_0 W_0 \qquad |W_0|^2 \simeq \mathcal{E}(t,\xi)$$

stabilization property

$$\int_0^{t_\xi} |\tilde{R}_0|\, d\tau \le C|\xi| \int_0^{t_\xi} |a(\tau) - a_0|\, d\tau \le C|\xi|(1+t_\xi)^\alpha \le CK$$

$$(1+t_\xi)^\alpha|\xi| = K$$

Estimate in Z_H under the stabilization property

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0 \quad \Rightarrow \quad \partial_t W_m = \tilde{R}_m W_m \qquad |W_m|^2 \simeq \mathcal{E}(t,\xi)$$

$$|\tilde{R}_m| \le C|\xi|^{-(m-1)}(1+t)^{-m\beta}$$

$$\int_{t_\xi}^{t} |\tilde{R}_m|\, d\tau \le C|\xi|^{-(m-1)}(1+t_\xi)^{-m\beta+1} = CK^{-m+1}(1+t_\xi)^{\alpha(m-1)+\alpha-m\beta+1} \le CK$$

$$\beta \ge \alpha + \frac{1-\alpha}{m}$$

$$\alpha = \kappa\beta$$

Theorem ([H4]). $a(t) \in C^m([0,\infty)), 0 < a_0 \le a(t) \le a_1,$

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le C(1+t)^\alpha \quad (0 \le \alpha < 1),$$

$$\left|a^{(k)}(t)\right| \le C_k(1+t)^{-k\beta} \quad (k = 1, \cdots, m),$$

$$\begin{cases} \beta \ge \alpha + \dfrac{1-\alpha}{m} \implies (GECL). \\ \beta < \alpha + \dfrac{1-\alpha}{m} \implies (GECL)\ \textit{is not valid in general.} \end{cases}$$

Corollary. $a(t) \in C^m([0,\infty)), 0 < a_0 \le a(t) \le a_1,$

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le C\Theta(t), \quad \lim_{t\to\infty} \Theta(t)/t = 0,$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)} \left(\frac{\Theta(t)}{1+t} \right)^{\frac{1}{m}} \right)^k \quad (k = 1, \cdots, m),$$

$$\implies (GECL).$$

We had supposed the condition $\Theta(2t)/\Theta(t) \leq C$ for in the paper [H], but actually such a condition is not necessary. The same property also holds for any monotone decreasing function $\Theta(t)$ satisfying $\lim_{t\to\infty} \Theta(t)/t = 0$.

Examples.

$$\Theta(t) = \frac{(1+t)}{(\log(e+t))^\gamma} \quad (\gamma > 0);$$

$$\Theta(t) = (1+t)^\alpha \quad (0 \leq \alpha < 1);$$

$$\Theta(t) = (\log(e+t))^\gamma \quad (\gamma > 0);$$

$$\Theta(t) = 1.$$

C^m property and upper bound of the energy

Theorem. $a(t) \in C^m([0,\infty))$, $0 < a_0 \le a(t) \le a_1$,

$$\int_0^t |a(\tau) - a_0|\, d\tau \le C\Theta(t), \quad \left(\lim_{t\to\infty} \frac{\Theta(t)}{t} \to 0\right),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{1+t} \right)^{\frac{1}{m}} \right)^k, \quad (k = 1, \cdots, m)$$

$$\Longrightarrow \quad \boxed{E(t) \le \exp\left(C\Theta(t)^{1-\beta}\right) E(0).}$$

Remark. The order of upper bound of the energy is optimal.

Example. $\Theta(t) = (1+t)^\alpha \ (0 < \alpha < 1)$,

$$\left|a^{(k)}(t)\right| \le C_k (1+t)^{-k\left(\alpha\beta - \frac{1-\alpha}{m}\right)}, \quad (k = 1, \cdots, m)$$

$$\Longrightarrow \quad \boxed{E(t) \le \exp\left(C(1+t)^{\alpha(1-\beta)}\right) E(0).}$$

Key of the proof

$$\Theta(t_\xi)^\beta|\xi| = K, \quad Z_H = \{t \geq t_\xi\}, \quad Z_\Psi = \{t < t_\xi\}.$$

In Z_H

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0 \quad \Longrightarrow \quad \partial_t W_m = \tilde{R}_m W_m \qquad |W_m|^2 \simeq \mathcal{E}(t,\xi)$$

$$\tilde{R}_m \in S_m^{-(m-1)} \quad \Longrightarrow \quad \int_{t_\xi}^t |\tilde{R}_m|\, d\tau \leq C|\xi|^{-m+1}\Theta(t)^{-m\beta+1} \leq C\Theta(t)^{1-\beta};$$

In Z_Ψ

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v = 0 \quad \Longrightarrow \quad \partial_t W_0 = \tilde{R}_0 W_0 \qquad |W_0|^2 \simeq \mathcal{E}(t,\xi)$$

$$\int_0^t |\tilde{R}_0|\, d\tau \leq \int_0^t |\xi||a(\tau) - a_\infty|\, d\tau \leq C|\xi|\Theta(t) \leq C\Theta(t)^{1-\beta} \quad (0 < t \leq t_\xi).$$

C^∞ property for GECL

Theorem. $a(t) \in C^\infty([0,\infty))$, $0 < a_0 \le a(t) \le a_1$,

$$\int_0^t |a(\tau) - a_0|\, d\tau \le C\Theta(t), \quad \left(\lim_{t\to\infty} \frac{\Theta(t)}{t} \to 0\right), \quad \left|a^{(k)}(t)\right| \le C_k \Theta(t)^{-k\beta} \quad (k = 1, 2, \cdots)$$

$$\Longrightarrow \begin{cases} E(t) \le \exp\left(C\Theta(t)^{1-\beta_0}\right) E(0) \quad (\forall \beta_0 > \beta); \\ E(t) \le \exp\left(C\Theta(t)^{1-\beta_1}\right) E(0) \quad \text{doesn't hold in general for any } \beta_1 < \beta. \end{cases}$$

Examples. $\Theta(t) = (1+t)^\alpha \quad (0 \le \alpha < 1)$

$$\left|a^{(k)}(t)\right| \le C_k (1+t)^{-k\alpha\beta} \quad (k = 1, 2, \cdots)$$

$$\Longrightarrow \quad E(t) \le \exp\left(C(1+t)^{\alpha(1-\beta_0)}\right) E(0)$$

$$\left|a^{(k)}(t)\right| \le C_k (1+t)^{-k\alpha_0} \quad ({}^\exists \alpha_0 > \alpha,\ k = 1, 2, \cdots) \Longrightarrow \text{(GECL)}$$

3. Strictly hyperbolic equations with non-Lipschitz coefficient

$$\textbf{(1)} \cdots \quad \begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

$$0 < a_0 \le a(t) \le a_1, \quad a(t) \in C^m((0,T]), \quad a(t) \notin Lip.([0,T]).$$

We shall consider the stabilization of the solution to (1) from the point of view of well-posedness.

<u>Gevrey class</u> $(s > 1)$

$$f(x) \in \gamma^{(s)} \iff \left|\partial_x^\alpha f(x)\right| \le C\ {}^\exists\rho^{|\alpha|}|\alpha|!^s \iff \left|\hat{f}(\xi)\right| \le \exp\left(-C\langle\xi\rangle^{\frac{1}{s}}\right)$$

$$C^\omega\ (\mathit{realanalytic}) \underset{(s\to 1)}{\Longleftarrow} \gamma^{(s)} \underset{(s\to\infty)}{\Longrightarrow} C^\infty\ (\text{or } H^\infty)$$

Well-posedness

(1) is well-posed in H

$$\iff (\nabla u_0, u_1) \in H \times H \;\Rightarrow\; {}^{\exists!}u(t,x) \;\text{ s.t. }\; \mathcal{E}_H(t,\xi) < \infty.$$

Examples.

L^2 well-posed $\iff \mathcal{E}(t,\xi) < C\mathcal{E}(T,\xi).$

H^s well-posed $\iff \mathcal{E}(t,\xi) < C\langle\xi\rangle^{2s}\mathcal{E}(T,\xi).$

C^∞ well-posed $\iff \mathcal{E}(t,\xi) < C\langle\xi\rangle^{{}^\exists M}\,\mathcal{E}(T,\xi).$

$\gamma^{(s)}$ well-posed $\iff \mathcal{E}(t,\xi) < \exp\left({}^\exists\rho\langle\xi\rangle^{\frac{1}{s}}\right)\,\mathcal{E}(T,\xi).$

Background

[CDS] $a(t) \in C^{\alpha}([0,T])\ (0 < \alpha < 1)$

$$\begin{cases} (1)\ \textit{is}\ \gamma^{(s)}\ \textit{well-posed for}\ s < 1/(1-\alpha); \\ (1)\ \textit{is not}\ \gamma^{(s')}\ \textit{well-posed for}\ \ s' > 1/(1-\alpha)\ \textit{in general.} \end{cases}$$

Motivation

We are interested in the stability of the solution in L^2, C^{∞} and $\gamma^{(s)}$ corresponding to the singularities of non-Lipschitz continuous coefficient. In particular, we focus the effects from *one point singularity* of the coefficient.

Probably, if the coefficient is non-Lipschitz only one point and sufficiently smooth on the other points, then we should expect the well-posedness of (1) in a better class than the case that the coefficient has several singular points.

L^2 and C^∞ well-posedness on C^1 property

Theorem. ([CDK])

$a(t) \in C^1((0,T]),$

$|a'(t)| \le Ct^{-\beta} \quad (\beta < 1) \implies (1)$ *is* L^2 *well-posed*;

$|a'(t)| \le Ct^{-1} \implies (1)$ *is* C^∞ *well-posed*;

$|a'(t)| \le Ct^{-\beta} \quad (\beta > 1) \implies (1)$ *is* $\gamma^{(s)}$ *well-posed for* $\dfrac{\beta}{\beta - 1}$.

Sketch of the proof

$\beta < 1$

$$\partial_t\, \mathcal{E}(t,\xi) = a(t)a'(t)|\xi|^2|v(t,\xi)|^2 \le Ct^{-\beta}\mathcal{E}(t,\xi)$$

$$\Longrightarrow \quad \mathcal{E}(t,\xi) \le \exp\left(C\int_0^T \tau^{-\beta}\, d\tau\right)\mathcal{E}(T,\xi) \le C\mathcal{E}(T,\xi)$$

$\beta \geq 1$

Zones: $t_\xi \langle \xi \rangle^\beta = K,\ Z_H := \{t_\xi \leq t \leq T\},\ Z_\Psi := \{0 \leq t < t_\xi\}$

In Z_H: $\partial_t\, \mathcal{E}(t,\xi) \leq C t^{-\beta} \mathcal{E}(t,\xi)$

$$\Rightarrow \quad \mathcal{E}(t,\xi) \leq \exp\left(C \int_{t_\xi}^{T} \tau^{-\beta}\, d\tau \right) \mathcal{E}(T,\xi)$$

$$\leq \begin{cases} \exp\left(C \log t_\xi^{-1} \right) \mathcal{E}(T,\xi) \leq C \langle \xi \rangle^{C} \mathcal{E}(T,\xi) & (\beta = 1) \\ \exp\left(C t_\xi^{-\beta+1} \right) \mathcal{E}(T,\xi) \leq \exp\left(C \langle \xi \rangle^{\frac{\beta-1}{\beta}} \right) \mathcal{E}(T,\xi) & (\beta > 1) \end{cases}$$

In Z_Ψ: $\mathcal{E}_0(t,\xi) := \dfrac{1}{2}\left(a_0^2 |\xi|^2 |v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right)$

$$\partial_t\, \mathcal{E}_0(t,\xi) \leq C |a_0 - a(t)|\, \langle \xi \rangle\, \mathcal{E}(t,\xi)$$

$$\Rightarrow \quad \mathcal{E}(t,\xi) \leq \exp\left(C \langle \xi \rangle \int_0^{t_\xi} |a_0 - a(t)|\, d\tau \right) \mathcal{E}(T,\xi) \leq \exp\left(C t_\xi \langle \xi \rangle \right) \mathcal{E}(T,\xi)$$

$$\leq \begin{cases} e^{CK} \mathcal{E}(T,\xi) \leq C \langle \xi \rangle^{C} \mathcal{E}(T,\xi) & (\beta = 1) \\ \exp\left(C \langle \xi \rangle^{\frac{\beta-1}{\beta}} \right) \mathcal{E}(T,\xi) & (\beta > 1) \end{cases}$$

Notes

We have no loss of regularity in $Z\Psi$ for $\beta=1$.
We observe similar situations as GECL on C^1 property.

Question. Can we apply C^2 and C^m property to such a problem?

L^2 and C^∞ well-posedness on C^2 property

Theorem. ([CDR], [H1, H2])

$$a(t) \in C^2((0,T]),$$

$$|a'(t)| \le Ct^{-1}, \quad |a''(t)| \le t^{-2} \implies (1) \textit{ is } L^2 \textit{ well-posed};$$

$$\left|a^{(k)}(t)\right| \le C\left(t^{-1}\log t^{-1}\right)^k \implies (1) \textit{ is } C^\infty \textit{ well-posed}.$$

Remark. The improvement of log-effect from C^2 property has no meaning in the scale of Gevrey class.

We notice some similarity of the conclusions from C^2 properties: "**generalized energy conservation law**" and "**well-posedness with non-Lipschitz coefficient**".
Thus we may also expect the **C^m property for the well-posedness with non-Lipschitz coefficients** corresponding to GECL.

C^2 property in Z_H

GECL $\quad (1+t_\xi)|\xi| = K, \quad \left|a^{(k)}(t)\right| \le C_k(1+t)^{-k} \quad (k=1,2),$

$$\int_{t_\xi}^{\infty} |R_2|\, d\tau \le C \int_{t_\xi}^{\infty} |\xi|^{-1}(1+\tau)^{-2}\, d\tau = K^{-1}.$$

$$\Updownarrow$$

Non-Lip. L^2 w.p. $\quad t_\xi\langle\xi\rangle = K, \quad \left|a^{(k)}(t)\right| \le C_k\, t^{-k} \quad (k=1,2),$

$$\int_{t_\xi}^{T} |R_2|\, d\tau \le C \int_{t_\xi}^{T} \langle\xi\rangle^{-1}\tau^{-2}\, d\tau = K^{-1}.$$

C^m property

GECL

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le \Theta(t), \quad \Theta(t_\xi)|\xi| = K,$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)} \left(\frac{\Theta(t)}{1+t} \right)^{\frac{1}{m}} \right)^k \quad (k = 1, \cdots, m),$$

$$\int_{t_\xi}^\infty |R_m|\, d\tau \le C \int_{t_\xi}^\infty |\xi|^{-m+1} \Theta(\tau)^{-m+1} (1+\tau)^{-1}\, d\tau \le K^{-m+1}.$$

⇕ **?**

Non-Lip. L^2 w.p.

$$\int_0^t |a(\tau) - a_0|\, d\tau \le \Theta(t), \quad \Theta(t_\xi)\langle\xi\rangle = K,$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k \quad (k = 1, \cdots, m),$$

$$\int_{t_\xi}^T |R_m|\, d\tau \le C \int_{t_\xi}^T \langle\xi\rangle^{-m+1} \Theta(\tau)^{-m+1} \tau^{-1}\, d\tau \le K^{-m+1}.$$

Theorem. (L^2 w.p. [CH]) $a(t) \in C^m((0,T]),\ 0 < a_0 \le a(t) \le a_1,$

$$\int_0^t |a(\tau) - a_0|\, d\tau \le \Theta(t),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k \quad (k = 1, \cdots, m),$$

$\Longrightarrow$ (1) *is L^2 well-posed*;

Example.

$$a(t) \in C^m((0,T]),\ \int_0^t |a(\tau) - a_0|\, d\tau \le C t^{\alpha} \quad (\alpha > 1),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(t^{-\alpha + \frac{\alpha - 1}{m}} \right)^k \quad (k = 1, \cdots, m) \Longrightarrow (1) \text{ is } L^2 \text{ well-posed.}$$

Any singularity of algebraic order at $t = 0$ is possible under suitable stabilization property and restrictions to higher order derivatives.

Gevrey well-posedness on C^m property

Theorem. (Upper bound of energy)

$$a(t) \in C^m([0,\infty)),\ \ 0 < a_0 \le a(t) \le a_1,\ \ \int_0^t |a(\tau) - a_0|\, d\tau \le \Theta(t),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{1+t} \right)^{\frac{1}{m}} \right)^k, \quad (k = 1, \cdots, m)$$

$$\Longrightarrow \quad E(t) \le \exp\left(C\Theta(t)^{1-\beta} \right) E(0).$$

$$\Theta(t_\xi)|\xi| = K \iff \Theta(t_\xi)\langle\xi\rangle = K$$

Theorem. (Gevrey well-posedness)

$$a(t) \in C^m([0,\infty)),\ \ 0 < a_0 \le a(t) \le a_1,\ \ \int_0^t |a(\tau) - a_\infty|\, d\tau \le \Theta(t),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k, \quad (k = 1, \cdots, m)$$

$$\Longrightarrow \quad \gamma^{(s)} \text{ well-posed for } s < \beta/(\beta - 1).$$

L^2 and Gevrey well-posedness on C^∞ property

Theorem. $a(t) \in C^\infty((0,T]),\ 0 < a_0 \le a(t) \le a_1,$

$$\int_0^t |a(\tau) - a_0|\, d\tau \le \Theta(t),$$

$$\left|a^{(k)}(t)\right| \le C_k \Theta(t)^{-k\beta_0}\ \ ({}^\exists\beta_0 < 1,\ k = 1, 2, \cdots) \implies L^2 \textit{ well-posed};$$

$$\left|a^{(k)}(t)\right| \le C_k \Theta(t)^{-k\beta}\ \ (\beta > 1,\ k = 1, 2, \cdots)$$

$$\implies \gamma^s \textit{ well-posed with } s < \frac{\beta}{\beta - 1}.$$

Examples. $\Theta(t) = t^\alpha\ \ (\alpha > 1)$

$$\left|a^{(k)}(t)\right| \le C_k\, t^{-k\alpha_0}\ \ ({}^\exists\alpha_0 < \alpha,\ k = 1, 2, \cdots) \implies L^2 \textit{ well-posed};$$

$$\left|a^{(k)}(t)\right| \le C_k\, t^{-k\alpha\beta}\ \ (\beta > 1,\ k = 1, 2, \cdots)$$

$$\implies \gamma^s \textit{ well-posed with } s < \frac{\beta}{\beta - 1}.$$

Remark. The conclusion of C^{∞} property dose not require any conditions to the constant C_k. However, if one want to consider the critical case, in particular, the L^2 well-posedness in the case that $\beta_0=1$, it may be required some conditions to the order of C_k. Namely, we should introduce some classification of C^{∞} functions, for instance, Gevrey class, real-analytic class, and so on.
Precisely, concerning GECL, the following problem can be proposed:

$\rho(t;s)$: positive and monotone increasing function satisfying

$$\lim_{t\to\infty}\frac{\rho(t;s)}{\Theta(t)^{\varepsilon}}=0\ \ ({}^{\forall}\varepsilon>0,\ {}^{\forall}s\ge 1),\quad \lim_{t\to\infty}\frac{\rho(t;s)}{\rho(t;s')}=0\ \ (s'>s).$$

$$\begin{cases}\int_0^t |a(\tau)-a_0|\,d\tau \le \Theta(t),\\ \left|a^{(k)}(t)\right| \le k!^s(\rho(t;s)\,\Theta(t))^{-k}\ \ ({}^{\forall}k\in\mathbb{N})\end{cases} \Longrightarrow \textbf{(GECL)}$$

Does such a class of functions $\{\rho(t;s)\}_{s\ge 1}$ exist?

On the Gevrey well-posedness for second order weakly hyperbolic equation with C^m coefficients

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4. Weakly hyperbolic equations degenerating at one point

$$\textbf{(1)} \cdots \begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

$$a(0) = 0, \quad a(t) > 0 \ (t > 0), \quad a(t) \in C^m((0,T]), \quad 0 < T \ll 1.$$

Background

[CJS] $a(t) \in C^{m,\alpha}([0,T])$: C^m-Hölder class $(m \in \mathbb{N} \cup \{0\},\ 0 < \alpha < 1)$

$$\begin{cases} (1) \textit{ is } \gamma^{(s)} \textit{ well-posed for } s < 1 + (m+\alpha)/2; \\ (1) \textit{ is not } \gamma^{(s')} \textit{ well-posed for } s' > 1 + (m+\alpha)/2 \textit{ in general.} \end{cases}$$

$a(t) \in C^{\alpha}([0,T])$, $\gamma^{(s)}$-well-posed:

[CDS] (strictly hyperbolic)

$$s < s_s := \frac{1}{1-\alpha}$$

$$\Longleftarrow \ s_s > s_w \ \Longrightarrow$$

[CJS] (weakly hyperbolic)

$$s < s_w := 1 + \frac{\alpha}{2}$$

Degeneracy of the coefficient should be set a singular property, which brings a loss of regularity log the solution.

Motivation. We want to describe the order of regularity loss of the solution by the following singular properties of the coefficients:

- ∘ Order of differentiability at $t > 0$: m;
- ∘ Stabilization property: Θ;
- ∘ Order of the singularities of higher order derivatives: β;
- ∘ Oder of the degeneracy: λ.

Known results for the models of one point degeneracy

$$a(t) = \lambda(t)\omega(t), \quad \lambda(0) = 0, \quad \lambda(t) > 0 \ (t > 0), \quad 0 < \omega_0 \le \omega(t) \le \omega_1,$$

$$\omega(t) \in C^m((0,T]), \quad \lambda(t) \in C^m((0,T]),$$

$$\left|\lambda^{(k)}(t)\right| \le C_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \quad (k = 1, \cdots, m), \qquad \Lambda(t) = \int_0^t \lambda(s)\, ds.$$

Theorem (Gevrey well-posedness; C[1] property)

$$|\omega'(t)| \le C_1 \frac{\lambda(t)}{\Lambda(t)^\beta} \quad (\beta > 1) \quad \Longrightarrow \quad (1) \ is \ \gamma^{(s)} \ w.p. \ for \ s < \beta/(\beta - 1).$$

Theorem (C$^\infty$ well-posedness [Y]; C^2 property)

$$\left|\omega^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)} \log \Lambda(t)^{-1}\right)^k \quad (k = 1, 2) \quad \Longrightarrow \quad (1) \ is \ C^\infty \ w.p.$$

Images of the generalization from strictly hyperbolic models to weakly hyperbolic models

$\gamma^{(s)}$ well-posedness for $s < \beta/(\beta - 1)$

$|a'(t)| \le C_1 t^{-\beta}$

$(1 \Longrightarrow \lambda(t))$

$(t \Longrightarrow \Lambda(t))$

$|\omega'(t)| \le C_1 \dfrac{\lambda(t)}{\Lambda(t)^\beta}$

C^∞ well-posedness on C^2 property

$$\left|a^{(k)}(t)\right| \le C_k \left(t^{-1} \log t^{-1}\right)^k \quad (k = 1, 2)$$

$(1 \Longrightarrow \lambda(t))$

$(t \Longrightarrow \Lambda(t))$

$$\left|\omega^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)} \log \Lambda(t)^{-1}\right)^k \quad (k = 1, 2)$$

$\gamma^{(s)}$ well-posedness on C^m property

Strictly hyperbolic

$$a(t)$$

$$\int_0^t |a(\tau) - a_0|\, d\tau \leq \Theta(t)$$

$$\left|a^{(k)}(t)\right| \leq C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k$$

$$\lambda(t)\omega(t)$$

$$\int_0^t \lambda(\tau)|\omega(\tau) - \omega_0|\, d\tau \leq \Theta(t)$$

$$\left|\omega^{(k)}(t)\right| \leq C_k \left(\frac{\lambda(t)}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{\Lambda(t)} \right)^{\frac{1}{m}} \right)^k$$

Weakly hyperbolic

(1) $$\begin{cases} (\partial_t^2 - \lambda(t)^2\omega(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in [0,T] \times \mathbb{R}^n, \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

$$\lambda(0) = 0, \quad \lambda(t) > 0 \ (t > 0), \quad 0 < \omega_0 \le \omega(t) \le \omega_1,$$

$$\omega(t) \in C^m((0,T]), \quad \lambda(t) \in C^m((0,T]),$$

$$\left|\lambda^{(k)}(t)\right| \le C_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \quad (k = 1, \cdots, m),$$

Examples of **λ**(t).

- $\lambda(t) = t^p, \quad \Lambda(t) \simeq t^{p+1} = t\lambda(t);$
- $\lambda(t) = \exp\left(-t^{-\alpha}\right), \quad \Lambda(t) \simeq t^{\alpha+1}\lambda(t);$

Theorem (**γ** [(s)] w.p. for weakly hyp. eq on C[m] property)

$$\int_0^t \lambda(\tau)|\omega(\tau)-\omega_0|\,d\tau \le \Theta(t), \quad \sup_{t\in(0,T)}\left\{\Lambda(t)\Theta(t)^{m(\beta-1)-1}\right\} < \infty,$$

$$\left|\omega^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}}\right)^k \quad (k=1,\cdots,m)$$

$\Longrightarrow$ (1) is $\gamma^{(s)}$ well-posed for $s < \frac{\beta}{\beta-1}$.

Example 1. $\lambda(t) = t^p, \quad \Theta(t) = t^{\delta+p+1}\ (\delta > 0), \quad m \ge \frac{1}{(\beta-1)(\delta+p+1)},$

$$\frac{\lambda(t)}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}} \simeq t^{p-\beta(\delta+p+1)+\frac{\delta}{m}} = t^{p-\beta(p+1)-\delta\left(\beta-\frac{1}{m}\right)} = \begin{cases} t^{p-\beta(p+1)-\delta(\beta-1)} & (m=1) \\ t^{-1-\delta\left(1-\frac{1}{m}\right)} & (\beta=1) \\ t^{p-\beta(p+1)} & (\delta=0) \end{cases}$$

Theorem (C^∞ w.p. for weakly hyp. eq on C^m property)

$$\int_0^t \lambda(\tau)|\omega(\tau)-\omega_0|\,d\tau \le \Theta(t), \quad \sup_{t\in(0,T)}\left\{\Lambda(t)\Theta(t)^{-1}\left(\log\Theta(t)^{-1}\right)^{-m}\right\} < \infty,$$

$$\left|\omega^{(k)}(t)\right| \le C_k\left(\frac{\lambda(t)}{\Theta(t)}\log\Theta(t)^{-1}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}}\right)^k \quad (k=1,\cdots,m)$$

⟹ (1) is C^∞ well-posed.

Example 1. $\lambda(t)=t^p, \quad \Theta(t)=t^{p+1}\left(\log t^{-1}\right)^{-r}, \quad m\ge r,$

$$\frac{\lambda(t)}{\Theta(t)}\log\Theta(t)^{-1}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}} \simeq t^{-1}\left(\log t^{-1}\right)^{r+1-\frac{r}{m}}$$

Example 2. $\lambda(t)=\exp\left(-t^{-\alpha}\right), \quad \Theta(t)=t^{q+\alpha+1}\lambda(t), \quad q\le\alpha m,$

$$\frac{\lambda(t)}{\Theta(t)}\log\Theta(t)^{-1}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}} \simeq t^{-2\alpha-1-q+\frac{q}{m}}$$

([Tarama (1995)] considered the optimality in the case q=0)

Key ideas for the proof for the Gevrey well-posedness

- Stabilization property : $\displaystyle\int_0^t \lambda(\tau)|\omega(\tau)-\tau_0|\,d\tau = \Theta(t);$
- Introduction of three zones:

$$Z_H := \{t \ge t_2(\xi)\},\quad Z_M := \{t_1(\xi) \le t < t_2(\xi)\},\quad Z_\Psi := \{0 \le t < t_1(\xi)\},$$

$$\boxed{\Lambda(t_1)^\beta\langle\xi\rangle = N,\quad \Theta(t_2)^\beta\langle\xi\rangle = N.}$$

In the respective zones, we introduce the following approximation of the coefficient:

$$\tilde{a}(t,\xi) = \begin{cases} \omega_0 & \text{in } Z_\Psi; \\ \omega_0\lambda(t) & \text{in } Z_M; \\ \lambda(t)\omega(t) & \text{in } Z_H. \end{cases}$$

Then, the stabilization property performs essentially in Z_M.

Remark. Introduction of the new zone Z_M has no meaning if no stabilization property is introduced.

5. GECL for strictly hyperbolic equations with increasing propagation speed

Previous considerations

$$\partial_t^2 u - a(t)^2 \Delta u = 0$$

Strictly hyp. eq., (GECL);

$(t \to \infty)$

$1 \to \lambda(t)$, $\lambda(\infty) = \infty$ ⇢

$$\partial_t^2 u - (\lambda(t)\omega(t))^2 \Delta u = 0$$

Strictly hyp. eq., with increasing coeff. (GECL);

$(t \to \infty)$

$\Theta(t_\xi)|\xi| = K$ $(t_\xi \gg 1)$ ⇓ $\Theta(t_\xi)\langle\xi\rangle = K$ $(\langle\xi\rangle \gg 1)$

$\Theta(t_\xi)|\xi| = K$, $\Lambda(t_\xi)|\xi| = K$ $(t_\xi \gg 1)$ ⇡ $\Theta(t_\xi)\langle\xi\rangle = K$, $\Lambda(t_\xi)\langle\xi\rangle = K$ $(\langle\xi\rangle \gg 1)$

$$\partial_t^2 u - a(t)^2 \Delta u = 0$$

Strictly hyp. eq., (well-posed);

$(t \to 0)$

$1 \to \lambda(t)$, $\lambda(0) = 0$ ⇒

$$\partial_t^2 u - (\lambda(t)\omega(t))^2 \Delta u = 0$$

Weakly hyp. eq., (well-posed);

$(t \to 0)$

Observations with examples

Model for C$^\infty$ (L^2) well-posedness: $\mathcal{E}(t,\xi) \le C\mathcal{E}(T,\xi) \quad (t \to 0)$

$$\lambda(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \le C\left(\lambda(T)^2|\xi|^2|v(T,\xi)|^2 + |v_t(T,\xi)|^2\right)$$

$\to$ 0 as t $\to$ 0

Model for GECL: $\mathcal{E}(t,\xi) \le C\mathcal{E}(0,\xi) \quad (t \to \infty)$

$$\lambda(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \le C\left(\lambda(0)^2|\xi|^2|v(0,\xi)|^2 + |v_t(0,\xi)|^2\right)$$

$\to \infty$ as t $\to \infty$

We may expect from the GECL above that the following decay estimate holds:

$$|\xi|\,|v(t,\xi)| \lesssim \mathcal{O}(\lambda(t)^{-1})$$

GECL with increasing propagation speed on C^m property

Theorem

$$\sup_{t\in(0,T)} \{\Theta(t)\Lambda(t)^{-\varepsilon}\} < \infty \ ({}^{\exists}\varepsilon \in (0,1)) \ , \Theta(2t)/\Theta(t) \le C,$$

$$\int_0^t \lambda(\tau)|\omega(\tau)-\omega_0|\,d\tau \le \Theta(t), \quad \left|\omega^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Theta(t)}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}}\right)^k \quad (k=1,\cdots,m)$$

$\Longrightarrow$ **(GECL)**

Remark. The properties of generalized energy conservation law with unbounded propagation speed is important, in particular, from the point of view to the application of "energy decay property for the wave equation with signed oscillating dissipation" and "GECL for Klein-Gordon type equation with oscillating signed mass". Probably, one can consider such problems only applying C^m property with stabilization property.

6. Proposal some open problems

(All the problems blow are supposed to consider on C^m and stabilization property.)

A. Farther problems for the models of wave equations:

A.1. L^p - C^q decay estimate with time dependent propagation speed

- Such problem was considered on C^2 property by [RS], [RY].
- Probably, we can prove a natural property as a generalization of previous result. But one should check the effect of stabilization property in the consideration of stationary phase method.

A.2. Well-posedness for time and space variable depending coefficient

- [HR] will be a hint to consider such a problem.
- $\boldsymbol{\gamma}^{(s)}$ well-posedness and weakly hyperbolic problem should be also considered.

A.3. GECL for time and space variable depending coefficient

- The global estimate of the lower order terms from commentator will be difficult.

B. Generalization to p-evolution model:

$$\left(D_t^2 - \sum_{k=0}^{2p} a_k(t) D_x^k\right) u = 0;$$

$$\int_0^t |a_k(\tau) - a_{k,0}|\, d\tau \le C_{k,0}\, t^{\alpha_k}, \quad |D_t^t a_k(t)| \le C_{k,l}\, t^{-\beta_{k,l}} \ (l = 1, \cdots, m)$$

B.1. L^2, C^∞ and $\gamma^{(s)}$ well-posedness for non-critical cases

- Dependences among the indexes α_k, β_{kl}, m and the Gevrey order s for the well-posedness will be described clearly.
- L^2 and C^∞ well-posedness on C^2 property has already considered in [CHR])

B.2. Weakly p-evolution equation

- Both the cases: oscillating and without oscillating coefficients are open.
- The property will be described smartly for non-critical cases.
- We will meet serious (but interesting) problems in the critical case.

B.3. GECL for p-evolution equations

C. Klein-Gordon equations with oscillating mass:

$$\left(\partial_t^2 - \Delta + m(t)^2\right) u = 0;$$

C.1. GECL for Klein-Gordon equations with oscillating mass

- We could prove only non-critical case, non-oscillating case or undetermined sign mass if we use by the previous arguments.
- If the mass decays as $t \to \infty$, then the relation between the oscillation and the decay order will be interesting.

C.2. L_p-L_q decay estimate

C.3. The analysis near the singular point to the model of blows up mass

- [DKR] considering such a model, probably by using C^2 approach.

D. Levi condition with oscillating coefficients:

- [HR2] may be a hit, but some crucial problem will appear.

E. Sophistication of C^∞ property:

Expected difficulty and impact

difficult ---- normal ---- easy : unclear

A.1. L^p - L^q decay estimate with time dependent propagation speed

A.2. Well-posedness for time and space variable depending coefficient

A.3. GECL for time and space variable depending coefficient

B.1. L^2, C^∞ and $\gamma^{(s)}$ well-posedness for non-critical cases

B.2. Weakly p-evolution equation

B.3. GECL for p-evolution equations

C.1. GECL for Klein-Gordon equations with oscillating mass

C.2. L^p-L^q decay estimate

C.3. The analysis near the singular point to the model of blows up mass

D. Levi condition with oscillating coefficients

E. Sophistication of C^∞ property

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