On the well-posedness for second order weakly hyperbolic Cauchy problems under the influences of the regularity of the coefficients

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Generalized energy conservation law for the wave equations with variable propagation speed

(1)
$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

(2)
$$0 < a_0 \leq a(t) \leq a_1,$$

$$E(t) := \frac{1}{2}\Big(a(t)^2 \, \|\nabla u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \Big) \quad (\|\cdot\| = \|\cdot\|_{L^2})$$

$\underline{a(t) \equiv a_\infty > 0 \text{ (constant)}}$

$$E'(t) \equiv 0 \iff E(t) \equiv E(0) : \text{ energy conservation}$$

$\underline{a(t) \in C^1([0,\infty))}$

$$E'(t) = a'(t)\, a(t)\, \|\nabla u(t,\cdot)\|^2 \begin{cases} \le \dfrac{2|a'(t)|}{a(t)} E(t) \\ \ge -\dfrac{2|a'(t)|}{a(t)} E(t) \end{cases}$$

$$\implies E(t) \begin{cases} \le \exp\left(\int_0^t \frac{2|a'(\tau)|}{a(\tau)}\, d\tau\right) E(0) \\ \ge \exp\left(-\int_0^t \frac{2|a'(\tau)|}{a(\tau)}\, d\tau\right) E(0) \end{cases}$$

$\underline{a'(t) \in L^1((0,\infty))}$

$$C_0 E(0) \le E(t) \le C_1 E(0) :$$

Generalized Energy Conservation Law = GECL

Question. Doesn't GECL hold in general if $a'(t) \notin L^1((0,\infty))$?

$$E'(t)\begin{cases} \geq 0 \quad \text{for } a'(t) > 0 \quad \Rightarrow \quad (E(t) \nearrow) \\ \leq 0 \quad \text{for } a'(t) < 0 \quad \Rightarrow \quad (E(t) \searrow) \end{cases}$$

Can we make a consideration the sign of $a'(t)$?

$a'(t) \in L^1$ ⟹ **(GECL)**

⟸ ?

$|a'(t)| \leq C(1+t)^{-\beta}$

$\beta > 1$ ⟹ **(GECL)**

⟸ ?

(1) ⇨

$$\begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)\, v(t,\xi) = 0 \\ (v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi)) \end{cases}$$

$$v(t,\xi) = A_{1+}(t,\xi)\, e^{\int_0^t \phi_{1+}(\tau,\xi)} + A_{1-}(t,\xi)\, e^{\int_0^t \phi_{1-}(\tau,\xi)}$$

$$\phi_{1\pm}(t,\xi) = \pm i\, a(t)|\xi|$$

$|A_{1\pm}(t,\xi)| \leq C$ uniformly in $[0,\infty) \times \mathbb{R}^n$ for $\boxed{\beta > 1}$

Remark

$a(t)$: const. ⇨

$$\begin{cases} \int_0^t \phi_{1\pm}(\tau,\xi)\, d\tau = \pm iat|\xi| \\ A_{1\pm}(t,\xi) = \dfrac{a|\xi|v_0 \mp iv_1}{2a|\xi|} \end{cases}$$

$$v(t,\xi) = A_{2+}(t,\xi)\, e^{\int_0^t \phi_{2+}(\tau,\xi)} + A_{2-}(t,\xi)\, e^{\int_0^t \phi_{2-}(\tau,\xi)}$$

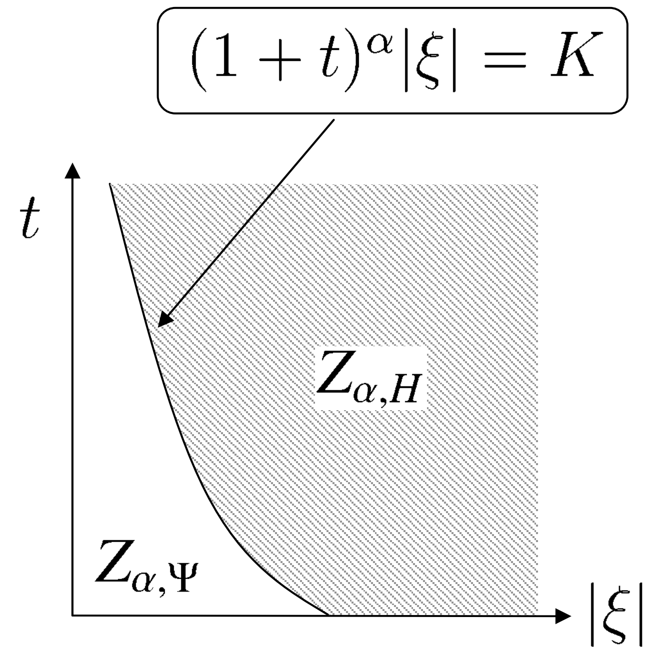
$$\phi_{2\pm}(t,\xi) = \pm i a(t)|\xi| + \frac{a'(t)}{2a(t)} = \phi_{1\pm}(t,\xi) + \frac{1}{2}(\log a(t))'$$

$$A_{2\pm}(t,\xi) = A_{2\pm}(t,\xi; a'(t), a''(t))$$

$$|a^{(k)}(t)| \le C_k(1+t)^{-k\beta}\ (k=1,2)$$

$$Z_{H,\alpha} := \{(t,\xi):\ (1+t)^{\alpha}|\xi| \gg 1\}$$

$|A_{2\pm}(t,\xi)| \le C$ uniformaly in $Z_{H,\alpha}$ for



$$\beta \ge \beta_2 := \frac{\alpha}{2} + \frac{1}{2}$$

$$\underline{Z_{\Psi,\alpha} := \{(t,\xi) : \ (1+t)^{\alpha}|\xi| \le K\}}$$

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le C(1+t)^{\alpha} \quad (\alpha \in [0,1))$$

(Stabilization property)

$$\mathcal{E}_0(t,\xi) := \frac{1}{2}\left(a_\infty^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right)$$

$$\Longrightarrow \mathcal{E}_0'(t,\xi) = \frac{1}{2}\left(a_\infty^2 - a(t)^2\right)|\xi|^2\Re\{v'\overline{v}\}$$

$$\le \frac{|a_\infty^2 - a(t)^2|}{2a_\infty}|\xi|\mathcal{E}_0(t,\xi) \le C|a_\infty - a(t)||\xi|\mathcal{E}_0(t,\xi)$$

$$\Longrightarrow \mathcal{E}_0(t,\xi) \le \mathcal{E}_0(0,\xi)\exp\left(C|\xi| \int_0^t |a_\infty - a(\tau)|\, d\tau\right)$$

$$\le C\mathcal{E}_0(0,\xi) \ \text{ in } \ Z_{\alpha,\Psi}$$

$$\left.\begin{array}{c} |a^{(k)}(t)| \le C_k(1+t)^{-k\beta} \ (k=1,2) \\ \int_0^t |a(\tau)-a_\infty|\,d\tau \le C(1+t)^{\alpha} \quad (\alpha \in [0,1)) \\ \beta \ge \beta_2 := \dfrac{\alpha}{2} + \dfrac{1}{2} \ (<1) \end{array}\right\} \Longrightarrow \textbf{(GECL)}$$

Remark [Reissig-Smith '05]: $\alpha = 1$ (no stabilization)

Example: $a(t) = 2 + \cos\left(\log(1+t)\right)$

$\underline{a(t) \in C^3([0,\infty))}$

$$v(t,\xi) = A_{3+}(t,\xi)\, e^{\int_0^t \phi_{3+}(\tau,\xi)} + A_{3-}(t,\xi)\, e^{\int_0^t \phi_{3-}(\tau,\xi)}$$

$$\phi_{3\pm}(t,\xi) = \phi_{2\pm}(t,\xi) \mp \underbrace{\frac{i\,\frac{(a')^2}{8a^3|\xi|}}{1-\frac{(a')^2}{16a^4|\xi|^2}}}_{\text{pure imaginary}} + \underbrace{\frac{\left(\frac{(a')^2}{16a^4|\xi|^2}\right)'}{2\left(\frac{(a')^2}{16a^4|\xi|^2}-1\right)}}_{\text{real}}$$

$$\left.\begin{array}{c} |a^{(k)}(t)| \le C_k(1+t)^{-k\beta}\ (k=1,2,3) \\ \int_0^t |a(\tau)-a_\infty|\,d\tau \le C(1+t)^{\alpha}\ \ (\alpha\in[0,1)) \\ \beta \ge \beta_3 := \frac{2\alpha}{3}+\frac{1}{3}\,(<\beta_2<1) \end{array}\right\} \Longrightarrow \textbf{(GECL)}$$

$\underline{a(t) \in C^m([0,\infty))}$

$$v(t,\xi) = A_{m+}(t,\xi)\, e^{\int_0^t \phi_{m+}(\tau,\xi)} + A_{m-}(t,\xi)\, e^{\int_0^t \phi_{m-}(\tau,\xi)}$$

$$\phi_{m\pm}(t,\xi) = \phi_{(m-1)\pm}(t,\xi) \mp \underbrace{i\theta_m(t,\xi)}_{\textbf{pure imaginary}} + \underbrace{\frac{\rho'_m(t,\xi)}{2(\rho_m(t,\xi)-1)}}_{\textbf{real}}$$

$$\left.\begin{array}{c} |a^{(k)}(t)| \le C_k(1+t)^{-k\beta}\ (k=1,\cdots,m) \\ \int_0^t |a(\tau)-a_\infty|\,d\tau \le C(1+t)^{\alpha}\ \ (\alpha \in [0,1)) \\ \beta \ge \beta_m := \alpha - \dfrac{\alpha-1}{m}\,(< \beta_{m-1} < 1) \end{array}\right\} \Longrightarrow \textbf{(GECL)}$$

Theorem ([H.])

$$a(t) \in C^m([0,\infty)),\ 0 < a_0 \le a(t) \le a_1,$$

$$\int_0^t |a(\tau) - a_0|\, d\tau \le \Theta(t), \quad \left(\lim_{t\to\infty} \frac{\Theta(t)}{t} \to 0\right),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{1+t}\right)^{\frac{1}{m}}\right)^k, \quad (\beta > 0,\ k = 1, \cdots, m)$$

$$\Longrightarrow \quad E(t) \le \exp\left(C\Theta(t)^{1-\beta}\right) E(0).$$

Well-posedness for weakly and strictly hyperbolic equations with non-Lipschitz coefficients

$$(3) \qquad \begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n, \\ (u(T,x), u_t(T,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

Gevrey class $(s > 1)$

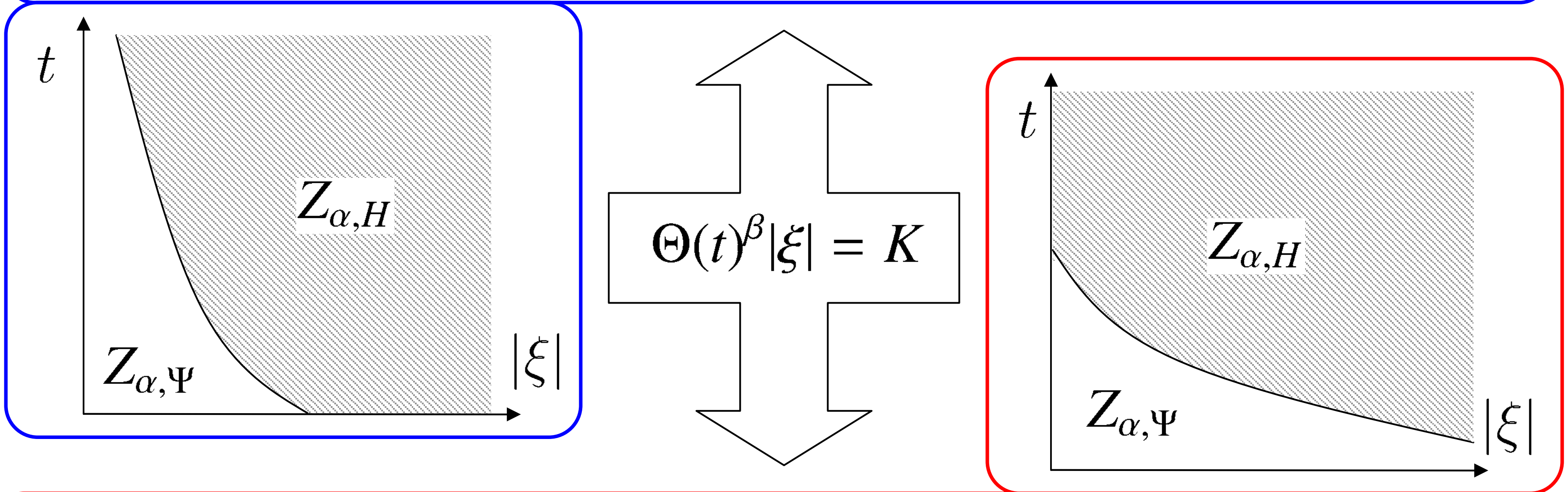
$$f(x) \in \gamma^{(s)} \iff \left|\partial_x^\alpha f(x)\right| \le C\rho^{|\alpha|}|\alpha|!^s \iff \left|\hat{f}(\xi)\right| \le \exp\left(-C\langle\xi\rangle^{\frac{1}{s}}\right)$$

$$C^\omega \ (realanalytic) \ \Longleftarrow \ \gamma^{(s)} \ \Longrightarrow \ C^\infty \ (\text{or } H^\infty,\ L^2)$$

$$(s \to 1) \qquad\qquad (s \to \infty)$$

$[0,\infty)\times\mathbb{R}^n_\xi$; energy estimate for **large** t with **small** $|\xi|$

$$\left|a^{(k)}(t)\right| \le C_k\left(\frac{1}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{1+t}\right)^{\frac{1}{m}}\right)^k, \quad \int_0^t |a_\infty - a(\tau)|\tau \le \Theta(t)$$



$(0,T]\times\mathbb{R}^n_\xi$: regularity estimate for **small** t with **large** $|\xi|$

$$\left|a^{(k)}(t)\right| \le C_k\left(\frac{1}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{t}\right)^{\frac{1}{m}}\right)^k, \quad \int_0^t |a_\infty - a(\tau)|\,d\tau \le \Theta(t)$$

Theorem. ([Cicognani – H.])

$$a(t) \in C^m((0,T]),\ \ 0 < a_0 \le a(t) \le a_1,$$

$$\int_0^t |a(\tau) - a_\infty|\, d\tau \le \Theta(t),$$

$$\left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k, \quad (k = 1, \cdots, m)$$

$$\Longrightarrow\ \gamma^{(s)} \text{ well-posed for } s < \beta/(\beta - 1).$$

<u>*Remark* (Equivalence of the estimates)</u>

$$E(t) \le \exp\left(C\Theta(t)^{1-\beta}\right) E(0) \iff \mathcal{E}(t,\xi) \le \exp\left(C\langle\xi\rangle^{\frac{\beta-1}{\beta}}\right)\mathcal{E}(T,\xi)$$

$$\Theta(t)^\beta|\xi| = K$$

Theorem. ([Colombini – Del Santo – Kinoshita '02])

$a(t) \in C^1((0,T]),$

$|a'(t)| \le Ct^{-\beta} \ \ (\beta < 1) \implies L^2$ *well-posed*;

$|a'(t)| \le Ct^{-1} \implies C^\infty$ *well-posed*;

$$\left|a'(t)\right| \le Ct^{-\beta} \ \ (\beta > 1)$$

$$\implies \gamma^{(s)} \text{ well-posed for } s < \frac{\beta}{\beta - 1}.$$

Remark

$$\Theta(t) = Ct \implies \left|a^{(k)}(t)\right| \le C_k \left(\frac{1}{\Theta(t)^\beta} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k \iff \left|a^{(k)}(t)\right| \le C_k t^{-k\beta}$$

Weakly hyperbolic equations

$$a(t) = \lambda(t)\omega(t), \quad \lambda(0) = 0, \quad \lambda(t) > 0\ (t > 0), \quad 0 < \omega_0 \le \omega(t) \le \omega_1,$$

$$\omega(t) \in C^m((0,T]), \quad \lambda(t) \in C^m((0,T]),$$

$$\left|\lambda^{(k)}(t)\right| \le C_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)}\right)^k \quad (k = 1, \cdots, m), \qquad \Lambda(t) = \int_0^t \lambda(s)\, ds.$$

Theorem (Classical result)

$$|\omega'(t)| \le C_1 \frac{\lambda(t)}{\Lambda(t)^\beta} \quad (\beta > 1) \quad \Longrightarrow \quad (1)\ \textit{is}\ \gamma^{(s)}\ \textit{w.p. for}\ s < \beta/(\beta - 1).$$

$\gamma^{(s)}$ well-posedness on C^m property

Strictly hyperbolic

$a(t)$

$$\int_0^t |a(\tau) - a_0| \, d\tau \leq \Theta(t)$$

$$\left|a^{(k)}(t)\right| \leq C_k \left(\frac{1}{\Theta(t)^{\beta}} \left(\frac{\Theta(t)}{t} \right)^{\frac{1}{m}} \right)^k$$

$\lambda(t)\omega(t)$

$$\int_0^t \lambda(\tau)|\omega(\tau) - \omega_0| \, d\tau \leq \Theta(t)$$

$$\left|\omega^{(k)}(t)\right| \leq C_k \left(\frac{\lambda(t)}{\Theta(t)^{\beta}} \left(\frac{\Theta(t)}{\Lambda(t)} \right)^{\frac{1}{m}} \right)^k$$

Weakly hyperbolic

Theorem ($\gamma^{(s)}$ w.p. for weakly hyp. eq with C^m coefficient)

$$\int_0^t \lambda(\tau)|\omega(\tau)-\omega_0|\,d\tau \le \Theta(t),$$

$$\sup_{t\in(0,T)}\left\{\Lambda(t)\Theta(t)^{m(\beta-1)-1}\right\} < \infty,$$

$$\left|\omega^{(k)}(t)\right| \le C_k\left(\frac{\lambda(t)}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}}\right)^k \quad (\beta>1,\ k=1,\cdots,m)$$

$\Longrightarrow$ (1) is $\gamma^{(s)}$ well-posed for $s < \frac{\beta}{\beta-1}$.

Example. $\lambda(t)=t^p,\ \ \Theta(t)=t^{\delta+p+1}\ (\delta>0),\ \ m\ge\frac{1}{(\beta-1)(\delta+p+1)}$,

$$\frac{\lambda(t)}{\Theta(t)^\beta}\left(\frac{\Theta(t)}{\Lambda(t)}\right)^{\frac{1}{m}} \simeq t^{p-\beta(\delta+p+1)+\frac{\delta}{m}} = t^{p-\beta(p+1)-\delta\left(\beta-\frac{1}{m}\right)} = \begin{cases} t^{p-\beta(p+1)-\delta(\beta-1)} & (m=1) \\ t^{p-\beta(p+1)} & (\delta=0) \end{cases}$$