C^m-theory of damped wave equations with stabilization

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1. Introduction

1.1. Problem

Cauchy problem for dissipative wave equation

$$\begin{cases} (\partial_t^2 - \Delta + 2b(t)\partial_t)\, u = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

$$E(t;u) = \frac{1}{2}\left(\|\nabla u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \right)$$

$$E'(t;u) = -2b(t)\|\partial_t u(t,\cdot)\|^2 \le 0 \quad (b(t) > 0)$$

Question. Does $E(t;u)$ decay or not?

If $E(t;u)$ decays, what about the decay order?

Example. $b(t) \in L^1(\mathbb{R}) \Rightarrow E'(t;u) \ge -4b(t)E(t;u)$

$\Rightarrow E(t;u) \ge E(0;u)\exp\left(-4\int_0^\infty b(s)\,ds\right)$ ***No decay!***

Known results (Matsumura, Mochizuki - Nakazawa, Uesaka, etc.)

Let $b_0 > 0$ and $E_0(t;u) = E(t;u) + \|u(t,\cdot)\|^2$.

$$b_0(1+t)^{-1} \le b(t) \le b_1 \Longrightarrow E(t;u) \lesssim (1+t)^{-1} E_0(0;u).$$

$$b(t) = b_0(1+t)^{-1} \Longrightarrow E(t;u) \lesssim (1+t)^{-\min\{2,2b_0\}} E_0(0;u).$$

$$b(t) = b_0(1+t)^{-1},\ 0 < b_0 < \tfrac{1}{2},\ E_0(0;u) < \infty,$$

$$\Longrightarrow \quad \exists C > 0,\ C^{-1}(1+t)^{-2b_0} \le E(t;u) \le C(1+t)^{-2b_0}.$$

(Wirth (2006))

Problem. Consider the perturbed dissipation:

$$b(t) = b_0(1+t)^{-1} + \sigma(t)$$

Which order of perturbation is allowed for the same energy estimate without the perturbation?

1.2. Generalized energy conservation

Cauchy problem for the wave equation

$$\begin{cases} (\partial_t^2 - a^2\Delta)\, u = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

$$E(t;a,u) = \frac{1}{2}\left(a^2 \|\nabla u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \right)$$

$$E'(t;a,u) = 0 \implies E(t;a,u) = E(0;a,u)$$

$\cdots$ ***Energy Conservation (EC)***

Let us generalize the propagation speed and total energy as follows:

$$a = a(t), \quad a(t) \in C^1([0,\infty)), \quad 0 < a_0 \le a(t) \le a_1$$

$$E(t;a,u) = E(t;a(t),u)$$

$$C^{-1}E(0;a(0),u) \le E(t;a(t),u) \le CE(0;a(0),u)$$

' ' ' Generalized Energy Conservation (GEC)

Proposition. $a'(t) \in L^1(\mathbb{R}_+) \Longrightarrow \text{(GEC)}.$

proof.

$$E'(0;a(t),u) = a'(t)a(t)\|\nabla u(t,\cdot)\|^2 = \frac{2a'(t)}{a(t)} \cdot \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2$$

$$\le \frac{2|a'(t)|}{a(t)} \cdot \frac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2$$

$$\le \frac{2|a'(t)|}{a(t)} E(t;a(t),u)$$

$$\Longrightarrow \quad E(t;a(t),u) \le E(0;a(0),u)\exp\left(\int_0^t \frac{2|a'(s)|}{a(s)}\,ds\right)$$

$$E(t;a(t),u) \ge E(0;a(0),u)\exp\left(-\int_0^t \frac{2|a'(s)|}{a(s)}\,ds\right)$$

Remark. (i) The following estimates have used:

$$-|a'(t)| \le a'(t) \le |a'(t)| \quad \text{and} \quad \tfrac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 \le E(t;a(t),u)$$

If we identify

$$|a'(t)| \ \text{with}\ a'(t) \quad \text{and} \quad \tfrac{1}{2}a(t)^2\|\nabla u(t,\cdot)\|^2 \ \text{with}\ \tfrac{1}{2}E(t;a(t),u),$$

then we have

$$\begin{aligned} E(t;a(t),u) &= E(0;a(0),u)\exp\left(\int_0^t \frac{a'(s)}{a(s)}\,ds\right) \\ &= \frac{a(t)}{a(0)}E(0;a(0),u) \end{aligned}$$

(ii) $a'(t) \in L^1(\mathbb{R}_+) \Longrightarrow |a(t)-a(s)| \to 0 \ (t,s\to\infty)$

Proposition. $a'(t) \in L^1(\mathbb{R}_+)$

$$\Longrightarrow |E(t;a(t),u) - E(s;a(s),u)| \to 0 \ (t,s\to\infty)$$

$$|E(t;a(t),u) - E(s;a(s),u)| \to 0 \ (t,s\to\infty)$$

$\cdots$ ***Asymptotically Free Energy (AFE)***

$$\text{(EC)} \implies \text{(AFE)} \implies \text{(GEC)}$$

Theorem A. (Reissig – Smith (2005)) $a(t) \in C^2([0,\infty))$,

$$|a'(t)| \lesssim t^{-1},\ |a''(t)| \lesssim t^{-2} \implies \text{(GEC)}$$

Theorem B. (H. (2007)) $a(t) \in C^m([0,\infty))$ $(m \geq 2)$,

$\int_0^t |a(s) - a_\infty|\,ds \lesssim t^\alpha,\ |a^{(k)}(t)| \lesssim t^{-k\alpha_m}$ $(k = 1,\cdots,m)$,

$$\alpha_m = \alpha - \tfrac{\alpha-1}{m} \implies \text{(GEC)}$$

$$E(t;a(t),u) \simeq \frac{a(t)}{a(0)} E(0;a(0),u)$$

Liouville transformation:

$$\tau = \int_0^t a(s)ds, \;\; a(t(\tau)) = \tilde{a}(\tau), \quad u(t(\tau), x) = w(\tau, x)$$

$$\left(\partial_t^2 - a(t)^2 \Delta\right) u(t, x) = 0$$

$$\boxed{\left(\partial_\tau^2 - \Delta + 2b(\tau)\partial_t\right) w(\tau, x) = 0, \;\; b(\tau) = \frac{\tilde{a}'(\tau)}{2\tilde{a}(\tau)}}$$

$$E(t; a(t), u) = \frac{1}{2}\, \tilde{a}(\tau)^2 \left(\|\nabla w(\tau, \cdot)\|^2 + \|\partial_\tau w(\tau, \cdot)\|^2 \right)$$

$$= \tilde{a}(\tau)^2 E(\tau; 1, w) \simeq E(\tau; w)$$

$$a(t) \in C^m \;\Leftrightarrow\; b(\tau) \in C^{m-1}, \;\; a(t) \le a_1 \;\Leftrightarrow\; \sup_t \left| \int_0^t b(s)ds \right| < \infty$$

$$\int_0^t |a(s) - a_\infty|\, ds \simeq \int_0^t \left| \int_\tau^\infty b(s)ds \right| d\tau$$

$$|a^{(k)}(t)| \simeq |b^{(k-1)}(t)|$$

$$\begin{cases} (\partial_t^2 - \Delta + 2b(t)\partial_t)\, u = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

Corollary. $b(t) \in C^m([0,\infty))\ (m \geq 1)$,

$$\sup_t \left| \int_0^t b(s)ds \right| < \infty, \quad \int_0^t \left| \int_\tau^\infty b(s)ds \right| d\tau \lesssim t^\beta \ (\beta < 1)$$

$$|b^{(k)}(t)| \lesssim t^{-(k+1)\beta_m} \ (k = 0, \cdots, m), \quad \beta_m = \beta + \tfrac{1-\beta}{m+1}$$

$$\Longrightarrow \text{(GEC)}; \quad E(t;u) \simeq \exp\left(-2\int_0^t b(s)ds \right) E(0;u)$$

Remark.

(i) $b(t)$ should be changing its sign;

(ii) (AFE) is not necessary to be satisfied;

(iii) $b(t)$ can be a non-L^1.

1.3. Energy decay for dissipative wave equations

$$0 < a_0 \leq a(t) \leq a_1 \quad \Longrightarrow \quad \left| \int_0^\infty b(s)ds \right| < \infty,$$

$$E(t;u) \simeq \exp\left(-2\int_0^t b(s)ds\right) E(0;u) \simeq E(0;w)$$

Energy decay problems cannot be handled!

If the following estimates are valid:

$$E'(t;u) = -2b(t)\|\partial_t u(t,\cdot)\|^2 \leq -\delta b(t) E(t;u) \quad (0 < {}^\exists\delta \leq 2)$$

then we have

$$E(t) \leq \exp\left(-\delta \int_0^t b(s)ds\right) E(0)$$

$$\int_0^t b(s)ds \to \infty \ (t \to \infty) \Longrightarrow \ \textbf{\textit{Energy decay}}$$

Positive monotone dissipation

$$b(t) = b_0(1+t)^{-1},\ b_0 > 0$$

Theorem C. (Wirth (2006)) $b_0 < 1/2,\ (u_0, u_1) \in H^1 \times L^2$

$$\Longrightarrow \begin{cases} E(t;u) \lesssim (1+t)^{-2b_0} E_0(0;u) \\ C^{-1} \le (1+t)^{2b_0} E(t;u) \le C \end{cases}$$

$$E_0(t;u) = E(t;u) + \|u(t,\cdot)\|^2, \quad C = C(E_0(0;u))$$

Remark. (i) $b(t)$ can be generalized to C^1 functions with some conditions for the monotonicity.

(ii) the decay order is given by $\exp(-2b_0 \int_0^t b(s)ds)$.

2. Main Results

2.1. Motivation

1) If decay order of the energy is given essentially by the ***Riemann integral*** of dissipation, then the effect of Riemann integrable perturbation of the dissipation should be neglected.

$$b(t) = b_0(1+t)^{-1} + \sigma(t), \ \ \sup_t \left| \int_0^t \sigma(s)ds \right| < \infty$$

$$\Longrightarrow \ E(t;u) \lesssim (1+t)^{-2b_0} E_0(0;u)$$

2) If one prove (GEC) for

$$\left(\partial_t^2 - a(t)^2 \Delta\right) u(t,x) = 0 \text{ with } \lim_{t\to\infty} a(t) = \infty,$$

then the energy estimate can be reduced into a decay estimate of dissipative wave equation after Liouville transformation.

2.2. Main Theorem

$$b(t) = b_0(1+t)^{-1} + \sigma(t), \quad \sup_t \left| \int_0^t \sigma(s)ds \right| < \infty$$

$$\sigma(t) \in C^m([0,\infty)) \quad (m \geq 1),$$

$$\int_0^t \left| \int_0^\tau \sigma(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta \quad (\beta < 1)$$

$$\left|\sigma^{(k)}(t)\right| \lesssim t^{-(k+1)\beta_m} \quad (k = 0, \cdots, m), \quad \beta_m = \beta + \frac{1-\beta}{m+1}$$

Main Theorem (H. - Wirth) $b_0 < 1/2$, $(u_0, u_1) \in H^1 \times L^2$

$$\Longrightarrow \begin{cases} E(t;u) \lesssim (1+t)^{-2b_0} E_0(0;u) \\ C^{-1} \leq (1+t)^{2b_0} E(t;u) \leq C \end{cases}$$

Remark.

$$\sup_t \left| \int_0^t \sigma(s)ds \right| < \infty$$ (Generalized zero mean condition)

$$\int_0^t \left| \int_0^\tau \sigma(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta \quad (\beta < 1)$$ (Stabilization property)

$$\left| \sigma^{(k)}(t) \right| \lesssim t^{-(k+1)\beta_m} \quad (k = 0, \cdots, m)$$ (C^m property)

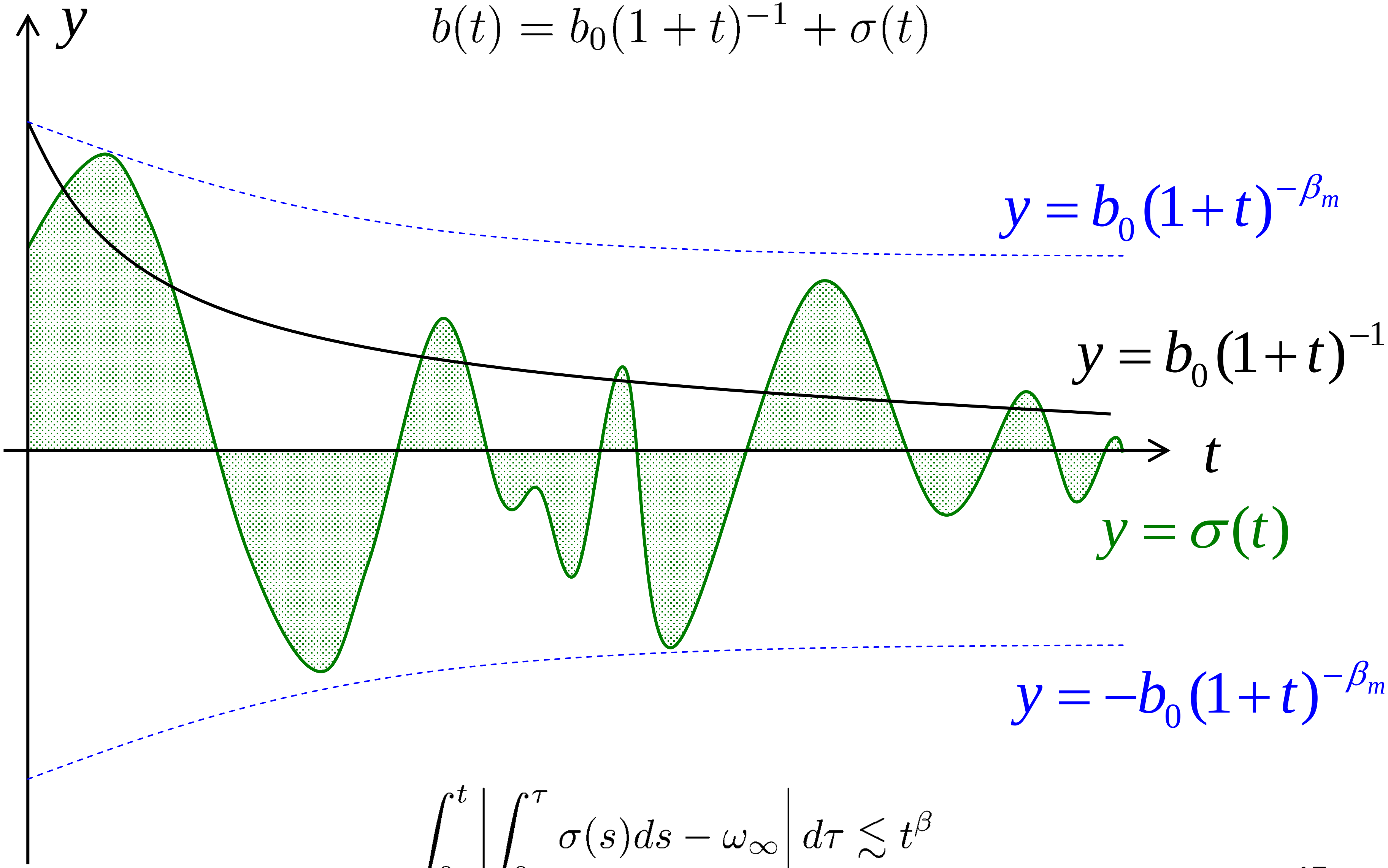
If $\beta = 1$, then the stabilization property is trivial, and β_m is independent of m.

β_m is monotone decreasing with respect to m, and

$$\lim_{m\to\infty} \beta_m = \lim_{m\to\infty} \beta + \frac{1-\beta}{m+1} = \beta$$

$$\left|\sigma^{(k)}(t)\right| \lesssim t^{-(k+1)\beta_m} \quad (k=0,\cdots,m), \quad \beta_m = \beta + \frac{1-\beta}{m+1}$$

$$b(t) = b_0(1+t)^{-1} + \sigma(t)$$



$$\int_0^t \left| \int_0^\tau \sigma(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta$$

Corollary. $\sigma(t) \in C^\infty([0,\infty))$,

$$\int_0^t \left| \int_0^\tau \sigma(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta \quad (\beta < 1)$$

$$\exists \beta_\infty > \beta, \quad \left|\sigma^{(k)}(t)\right| \leq C_k\, t^{-(k+1)\beta_\infty} \quad (k = 0, 1, \cdots),$$

$b_0 < 1/2$, $(u_0, u_1) \in H^1 \times L^2$

$$\Longrightarrow \begin{cases} E(t;u) \lesssim (1+t)^{-2b_0} E_0(0;u) \\ C^{-1} \leq (1+t)^{2b_0} E(t;u) \leq C \end{cases}$$

Stabilization property

$$\left(\partial_t^2 - a(t)^2\Delta\right) u = 0: \quad a_0 \le a(t) \le a_1 \;\Rightarrow\; \int_0^t |a(s) - a_\infty| ds$$

(Stabilization property for **bounded** propagation speed)

$$\left(\partial_t^2 - (\lambda(t)a(t))^2\Delta\right) u = 0, \quad \lim_{t\to\infty} \lambda(t) = \infty:$$

$$\Longrightarrow \int_0^t \lambda(s)|a(s) - a_\infty| ds,$$

(Stabilization property for **unbounded** propagation speed)

By Liouville transformation:

$$\tau(t) = \int_0^t \lambda(s)a(s)ds, \quad \tilde{\lambda}(\tau(t)) = \lambda(t), \quad \tilde{a}(\tau(t)) = a(t)$$

we have

$$\left(\partial_t^2 - (\lambda(t)a(t))^2\Delta\right) u = 0 \;\Leftrightarrow\; \left(\partial_\tau^2 - \Delta + 2b(\tau)\partial_\tau\right)\, w = 0$$

$$b(\tau) = b_0(1+\tau)^{-1} + \sigma(\tau) = \frac{(\tilde{\lambda}(\tau)\tilde{a}(\tau))'}{2\tilde{\lambda}(\tau)\tilde{a}(\tau)}$$

$$\tilde{\lambda}(\tau) = (1+\tau)^{2b_0} \implies \sigma(\tau) = \frac{\tilde{a}'(\tau)}{2\tilde{a}(\tau)}$$

$$\int_0^t \lambda(s)|a(s) - a_\infty| ds = \int_0^\tau \left| a(0) \exp\left(2\int_0^s \sigma(\mu)d\mu\right) - a_\infty \right| ds$$

$$\simeq \int_0^\tau \left| \int_0^s \sigma(\mu)d\mu - \omega_\infty \right| d\tau$$

3. Sketch of the proof

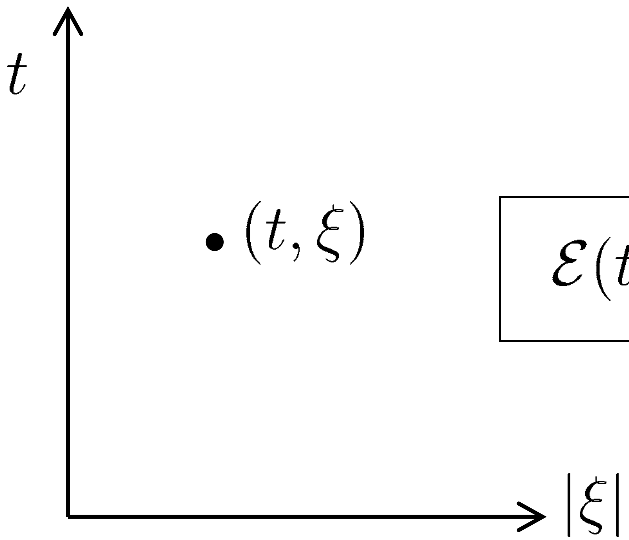
3.1. Zones in the phase space

Reduce our problem as follows by Fourier transformation:

$$\begin{cases} \left(\partial_t^2 + |\xi|^2 + 2b(t)\partial_t\right) v = 0, & (t,\xi) \in [0,\infty) \times \mathbb{R}^n, \\ (v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi)), & \xi \in \mathbb{R}^n. \end{cases}$$

Consider the microenergy of the solution:

$$\mathcal{E}(t,\xi) = \frac{1}{2}\left(\left|\xi| v(t,\xi)\right|^2 + \left|\partial_t v(t,\xi)\right|^2 \right)$$



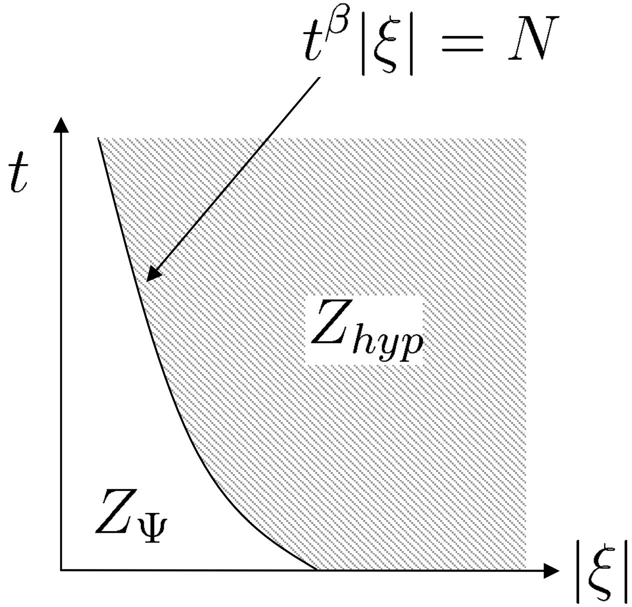
$$\boxed{\mathcal{E}(t,\xi) \lesssim (1+t)^{-2b_0} \left(\mathcal{E}(0,\xi) + |v(0,\xi)|^2\right)}$$

$a_0 \le a(t) \le a_1$ (Bounded propagation speed, (GEC))

$$\Longleftrightarrow \quad \sup_t \left| \int_0^t b(s)ds \right| < \infty \quad \text{(non-decay)}$$

$$\int_0^t \left| \int_0^\tau b(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta$$

N: large constant

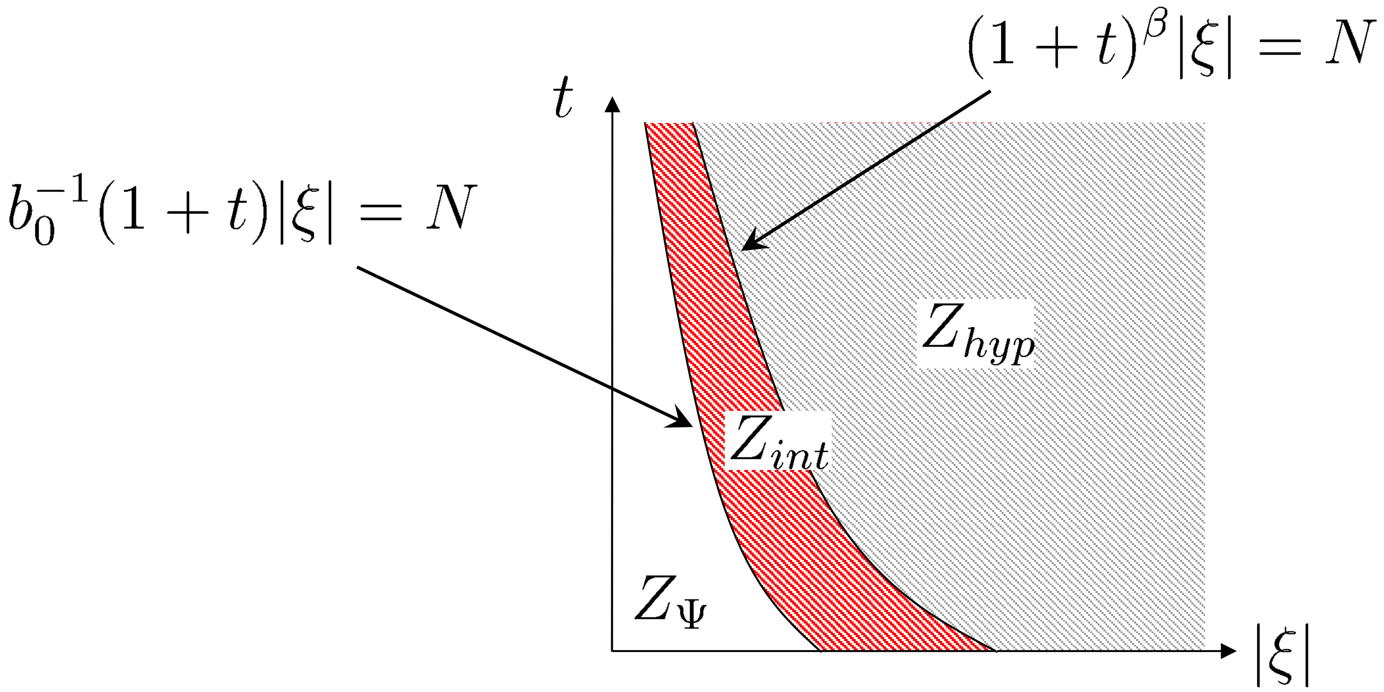


$$a_0\lambda(t) \le \lambda(t)a(t) \le a_1\lambda(t) \quad \text{(Increasing propagation speed)}$$

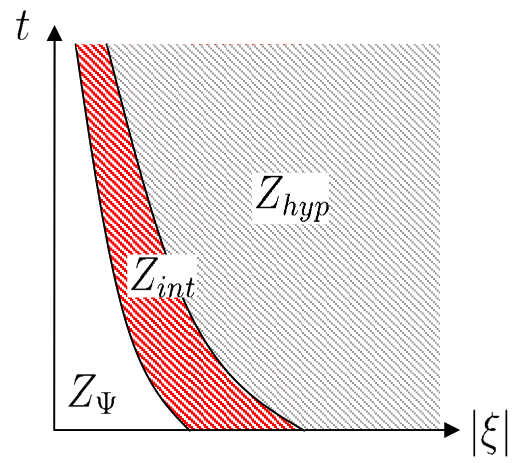
$$\Longleftrightarrow \int_0^t b(s)ds = \int_0^t b_0(1+s)^{-1}ds + \int_0^t \sigma(s)ds$$

$$= \log(1+t)^{b_0} + \int_0^t \sigma(s)ds$$

$$\int_0^t \left| \int_0^\tau b(s)ds - \omega_\infty \right| d\tau \lesssim t^\beta$$



We construct suitable approximate WKB solutions in respective zones.



$$\left(\partial_t^2 + |\xi|^2 + 2b(t)\partial_t\right) v = 0$$

$$\Downarrow$$

$$\partial_t V_1(t,\xi) = A_1(t,\xi)V_1(t,\xi)$$ (first order system)

$$V_1 = \begin{pmatrix} v_t + i|\xi|v \\ v_t - i|\xi|v \end{pmatrix} \quad A_1 = \begin{pmatrix} -b(t) + i|\xi| & -b(t) \\ -b(t) & -b(t) - i|\xi| \end{pmatrix}$$

$$|V_1(t,\xi)|^2 \simeq \mathcal{E}(t,\xi;v)$$

$$W_1 = \begin{pmatrix} e^{\int_0^t b(s)ds} & 0 \\ 0 & e^{\int_0^t b(s)ds} \end{pmatrix} V_1 \ = (1+t)^{b_0} e^{\int_0^t \sigma(s)ds} V_1$$

$$|W_1(t,\xi)|^2 \simeq (1+t)^{2b_0} |V_1(t,\xi)|$$

$$|W_1(t,\xi)|^2 \simeq |W_1(0,\xi)|^2 \iff (1+t)^{2b_0}|V_1(t,\xi)|^2 \simeq |V_1(0,\xi)|^2$$

$\cdots$ **decay estimate**

$$\partial_t W_1 = (\Lambda_1 + B_1)\, W_1$$

$$\Lambda_1 = \begin{pmatrix} i|\xi| & 0 \\ 0 & -i|\xi| \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -b(t) \\ -b(t) & 0 \end{pmatrix},$$

$$|W_1(t_1,\xi)| \lesssim |W_1(t_0,\xi)| \exp\left(\int_{t_0}^{t_1} |b(s)|ds\right)$$

$$\lambda_{1\pm} = \pm i|\xi|, \quad \beta_{1\pm} = -b(t)$$

Diagonalization in the hyperbolic zone Z_H

$$M_1 = \begin{pmatrix} 1 & \frac{\beta_{1+}}{\lambda_{1+}-\lambda_{1-}} \\ \frac{\beta_{1+}}{\lambda_{1+}-\lambda_{1-}} & 1 \end{pmatrix}, \quad W_2 = M_1^{-1} W_1 \quad \text{in } \; Z_H$$

$$\partial_t W_1 = (\Lambda_1 + B_1)\, W_1 \;\Leftrightarrow\; \partial_t W_2 = (\Lambda_2 + B_2)\, W_2$$

$$\lambda_{2\pm} = \pm i \left(|\xi| + \frac{3b^2}{4|\xi|} \right) - \frac{b'(t)b(t)}{4|\xi|^2}, \quad \beta_{2\pm} = -\frac{b(t)^3}{4|\xi|^2} \mp i \frac{b'(t)}{2|\xi|}$$

$$\Re\{\lambda_{2+} - \lambda_{2-}\} = 0, \quad \beta_{2+} = \overline{\beta_{2-}}, \quad \left| \int \Re\{\lambda_{2\pm}\}\, dt \right| \text{ is bounded in } Z_H$$

Generally, we have

$$\partial_t W_k = (\Lambda_k + B_k)\, W_k, \quad W_k = M_{k-1}^{-1} \cdots M_1^{-1} W_1, \quad |W_k| \simeq |W_1| \text{ in } Z_H$$

Moreover, if we introduce the symbol class:

$$S\{m_1, m_2\} = \left\{ \left|\partial_t^k \partial_\xi^\alpha a(t,\xi)\right| \le C_{k,\alpha} |\xi|^{m_1-|\alpha|} (1+t)^{-(k+m_2)\beta_m} \right\}$$

in the hyperbolic zone, then we have

$$B_k \in S\{-k, k+1\}.$$

Here we note that

$$S\{m_1 - k, m_2 + k\} \subset S\{m_1, m_2\} \quad (k \ge 0)$$

$$b^{(k)} \in S\{0, k+1\}$$

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