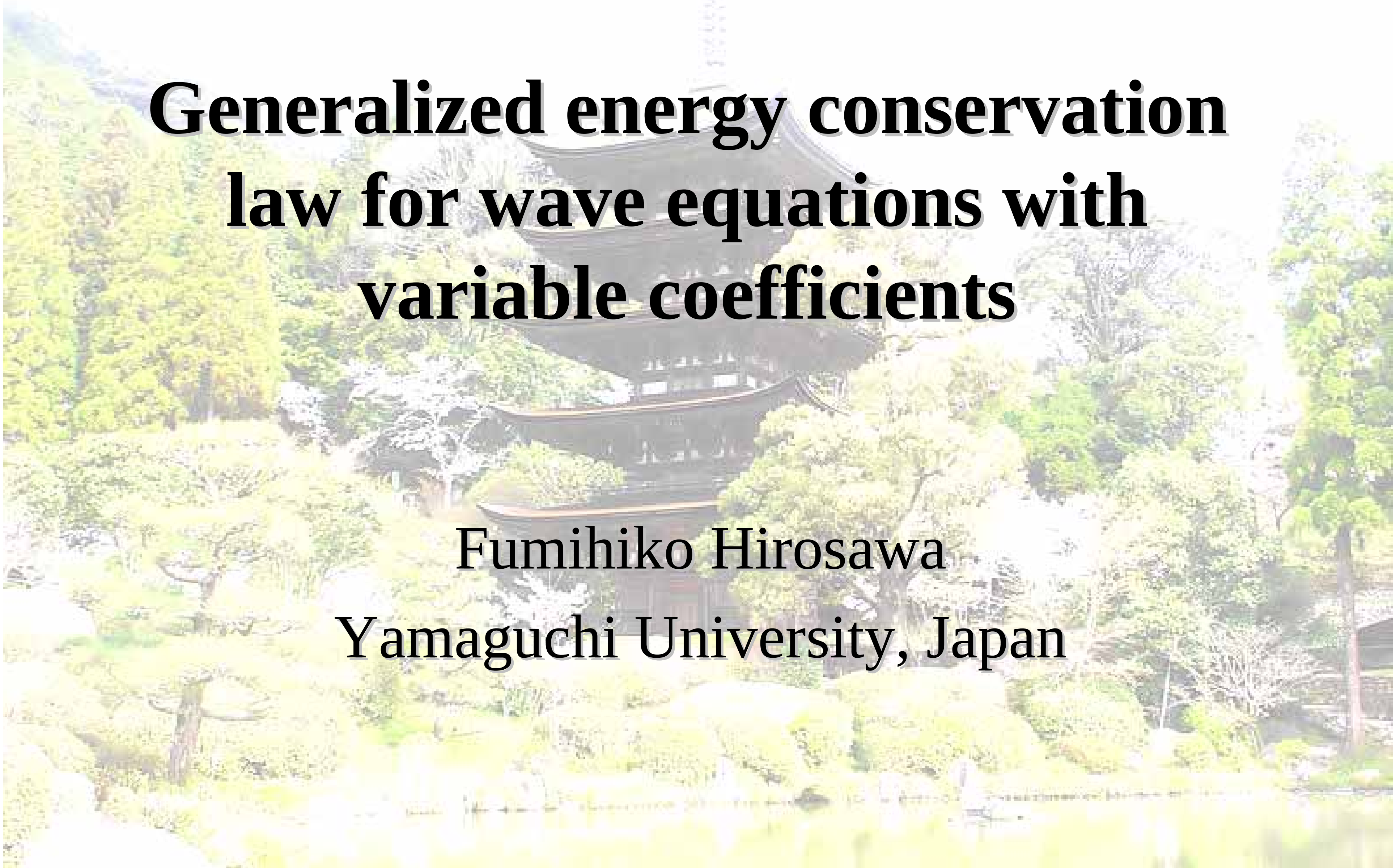

Generalized energy conservation law for wave equations with variable coefficients

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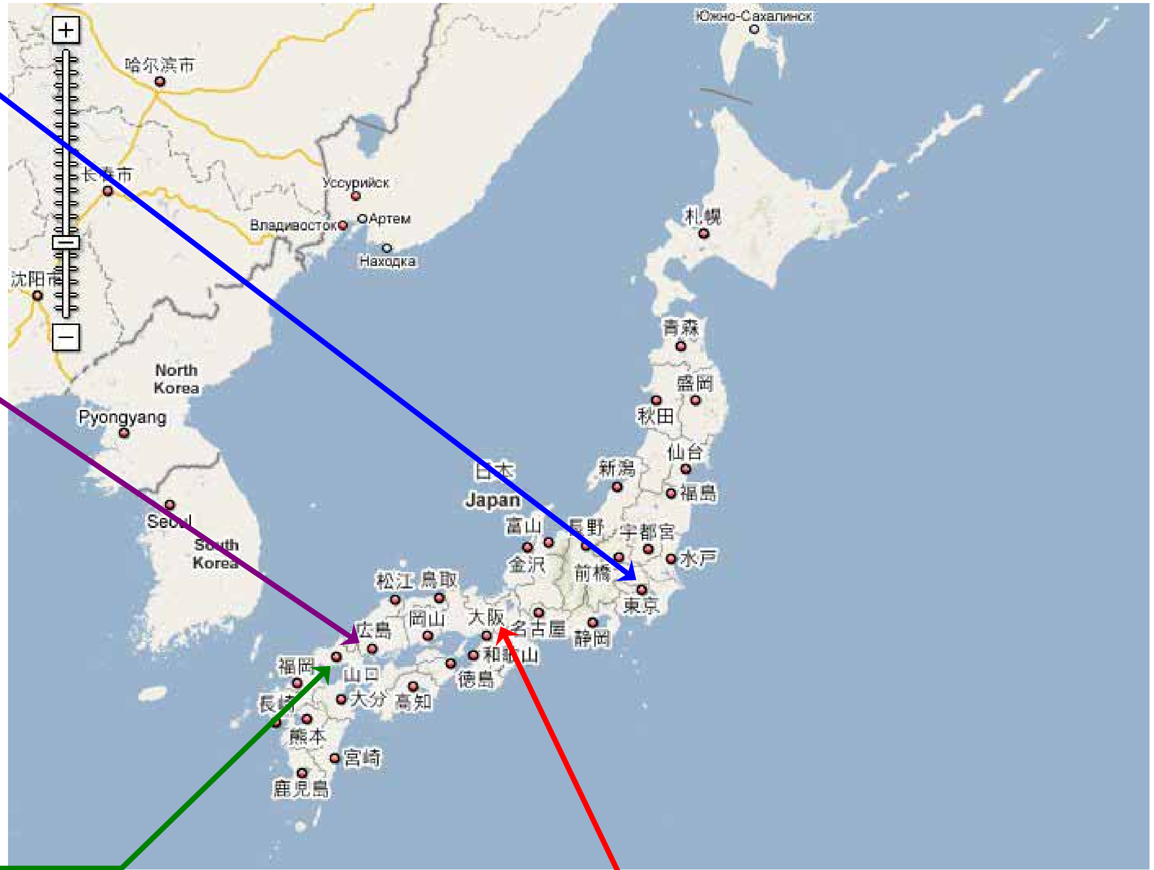

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Introduction

Kirchhoff equation (vibration of elastic string):

$$\text{(K)} \qquad \left(\partial_t^2 - \left(1 + \int_I |\partial_x u(t,x)|^2 dx\right) \partial_x^2\right) u(t,x) = 0$$

$$\Longrightarrow \quad \left(\partial_t^2 - a(t)^2 \partial_x^2\right) u(t,x) = 0$$

$$E_s(t) = \frac{1}{2}\left(a(t)^2 \|\partial u(t,\cdot)\|_s^2 + \|\partial_t u(t,\cdot)\|_s^2\right)$$

$$\begin{aligned} E_s'(t) &= a'(t) a(t) \|\partial u(t,\cdot)\|_s^2 \\ &= 2a(t) \Re(\partial \partial_t u(t,\cdot), \partial u(t,\cdot))_s \|\partial u(t,\cdot)\|_s^2 \\ &\le 2a(t) \|\partial_t u(t,\cdot)\|_{\frac{1}{2}} \|\partial u(t,\cdot)\|_{\frac{1}{2}} \|\partial u(t,\cdot)\|_s^2 \\ &\le C E_s(t)^2 \end{aligned}$$

$$E_s(t) \le (E_s(0) - Ct)^{-1} \quad \cdots \text{ time local estimate!}$$

Consider the following Cauchy problem:

$$\text{(C)} \begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n. \end{cases}$$

(A0) $$0 < a_0 \le a(t) \le a_1$$

[Colombini - De Giorgi - Spagnolo (1979)]

$a \in C^{\alpha}([0,\infty)) \;\Rightarrow\;$ (C) *is* γ^s *well-posed with* $s < \frac{1}{1-\alpha}$

$f(x) \in \gamma^s$ (Gevrey class of order *s*)

$$\Leftrightarrow \; |\hat{f}(\xi)| \le C \exp\left(-\rho|\xi|^{\frac{1}{s}}\right) \;\; (s > 1,\; \exists C > 0,\; \exists \rho > 0)$$

$$\forall s > \frac{1}{\alpha - 1},\; |(\hat{u}(0,\xi), \hat{u}_t(0,\xi))| \exp\left(\rho|\xi|^{\frac{1}{\tilde{s}}}\right) < \infty,\; \exists T > 0$$

$$\lim_{t \to T-0} |(\hat{u}(t,\xi), \hat{u}_t(t,\xi))| \exp\left(-\rho|\xi|^{\frac{1}{\tilde{s}}}\right) = \infty$$

$\underline{a(t) \in C^1([0,\infty))}$

$$E(t) = \frac{1}{2}\left(a(t)^2 \|\nabla u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \right)$$

$$a(t) \equiv const. \Rightarrow E(t) \equiv E(0) \quad \textit{(Energy Conservation)}$$

$$a(t) \not\equiv const. \Rightarrow E(t) \not\equiv E(0)$$

$$a'(t) > 0 \Rightarrow E'(t) \geq 0, \; a'(t) < 0 \Rightarrow E'(t) \leq 0$$

$$E'(t) = a'(t)\, a(t) \|\nabla u(t,\cdot)\|^2 \begin{cases} \leq \frac{2|a'(t)|}{a(t)} E(t) \\ \geq -\frac{2|a'(t)|}{a(t)} E(t) \end{cases}$$

$$\Longrightarrow E(t) \begin{cases} \leq \exp\left(\int_0^t \frac{2|a'(\tau)|}{a(\tau)}\, d\tau \right) E(0) \\ \geq \exp\left(- \int_0^t \frac{2|a'(\tau)|}{a(\tau)}\, d\tau \right) E(0) \end{cases}$$

$$\forall T > 0,\ \exists C = C_T > 0,\ \forall t \in [0, T]$$

$$C^{-1} E(0) \le E(t) \le C E(0)$$

(*Generalized Energy Conservation* = GEC)

$$a'(t) \in L^1((0,\infty)) \ \Rightarrow\ (GEC) \quad \text{unif. w.r.t. } t$$

Question: Doesn't (GEC) hold in general if $a'(t) \notin L^1$?

Can we take a cancellation of the oscillating energy due to the oscillating coefficient?

⇧

$$E'(t) \begin{cases} \ge 0 & \text{for } a'(t) > 0 \ \Rightarrow\ (E(t) \nearrow) \\ \le 0 & \text{for } a'(t) < 0 \ \Rightarrow\ (E(t) \searrow) \end{cases}$$

Remark

$$a(t) = 2 + \cos(2\pi\omega(t)),\ \ \omega'(t) \geq 0,\ \ \omega(0) = 0$$

$$\Longrightarrow \quad \int_0^T |a'(t)|dt \simeq \int_0^T |\omega'(t)|dt = \omega(T) :$$

number of oscillations on $[0, T]$

$$a'(t) \in L^1((0, \infty)) \ \Leftrightarrow \ \text{finite numbers of oscillations}$$

Question: Can (GEC) hold for infinitely oscillating coefficients?

Consider the following condition:

$$|a'(t)| \leq C(1 + t)^{-\beta}$$

$\beta > 1$ $\Longrightarrow$ (GEC)

$\Longleftarrow$?

Infinitely oscillating coefficients

Example. $a(t) = 2 + \cos\left((1+t)^p\right)$ $(p \le 0 \Leftrightarrow a'(t) \in L^1)$

$$p \le 0 \Leftrightarrow (GEC)$$

Example. ([Reissig-Smith (2005)])

$a(t) = 2 + \cos\left((\log(e+t))^\gamma\right)$ $(\gamma \le 0 \Leftrightarrow a'(t) \in L^1)$

$$\gamma \le 1 \Leftrightarrow (GEC)$$

Finite number of oscillations is not necessary for (GEC)!

Main purpose: Find the conditions to $a(t)$ for (GEC).

Theorem. ([Reissig-Smith (2005)])

$$\left.\begin{aligned} |a'(t)| &\le C(1+t)^{-1} \\ |a''(t)| &\le C(1+t)^{-2} \end{aligned}\right\} \Longrightarrow (GEC)$$

Remark.

(i) $|a'(t)| \le C(1+t)^{-1}$ is sharp for (GEC).

(ii) C^2 property of $a(t)$ is required.

(iii) Necessity of the condition to $a''(t)$ is an open problem.

Sketch of the proof.

$$\text{(C)} \qquad \begin{cases} (\partial_t^2 + a(t)^2|\xi|^2)\, v(t,\xi) = 0 \\ (v(0,\xi), v_t(0,\xi)) = (\hat{u}_0(\xi), \hat{u}_1(\xi)) \end{cases}$$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1, \quad V_1 = \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix} = \begin{pmatrix} \partial_t v + ia|\xi|v \\ \partial_t v - ia|\xi|v \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} \frac{a'}{2a} + ia|\xi| & 0 \\ 0 & \frac{a'}{2a} - ia|\xi| \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -\frac{a'}{2a} \\ -\frac{a'}{2a} & 0 \end{pmatrix}$$

$$\Longrightarrow \quad \partial_t |v_1|^2 = -\frac{a'}{a}\Re\{v_1\overline{v_1}\} \begin{cases} \le \frac{|a'|}{a}|v_1|^2 \\ \ge -\frac{|a'|}{a}|v_1|^2 \end{cases}$$

$$\Longrightarrow \quad |v_1(t,\xi)| \begin{cases} \le |v_1(t_0,\xi)| \exp\left(\int_{t_0}^t \frac{|a'(s)|}{a(s)} ds\right) \\ \ge |v_1(t_0,\xi)| \exp\left(-\int_{t_0}^t \frac{|a'(s)|}{a(s)} ds\right) \end{cases}$$

$$\partial_t V_2 = (\Phi_2 + B_2) V_2,$$

$$V_2 = \begin{pmatrix} v_2 \\ \overline{v_2} \end{pmatrix}, \quad v_2 = \frac{1}{1 - |\delta_1|^2} (v_1 - i \delta_1 \overline{v_1})$$

$$\Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & \overline{b_2} \\ b_2 & 0 \end{pmatrix}, \quad \delta_1 = i \frac{a'}{4a^2|\xi|}$$

$$\begin{aligned} \phi_2 =& \frac{1}{2} \partial_t \left(\log \left(\frac{a}{1 - |\delta_1|^2} \right) \right) + i \left(a|\xi| - \frac{2|\delta_1|^2}{1 - |\delta_1|^2} \right) \\ =& \phi_{2,\Re} + i \phi_{2,\Im} \end{aligned}$$

$$b_2 = \frac{1}{1 - \delta_1} \left(- \frac{(a')^3}{32 a^5 |\xi|^2} - i \left(\frac{a''}{4a^2|\xi|} - \frac{(a')^2}{2a^3|\xi|} \right) \right)$$

$$\partial_t |v_2|^2 = 2\phi_{2,\Re}|v_2|^2 + 2\Re\{b_2 v_2 \overline{v_2}\} \begin{cases} \leq 2(\phi_{2,\Re} + |b_2|)|v_2|^2 \\ \geq 2(\phi_{2,\Re} - |b_2|)|v_2|^2 \end{cases}$$

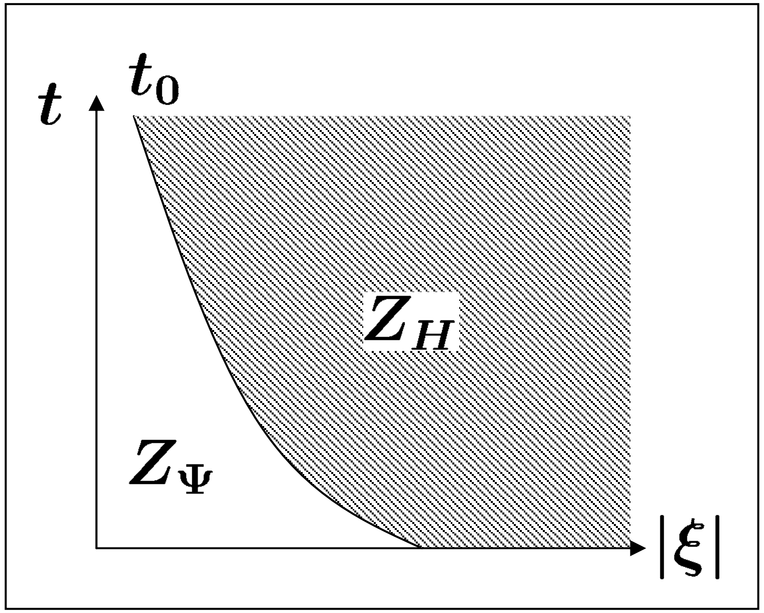
$$\Rightarrow \quad |v_2(t,\xi)|^2 \begin{cases} \leq |v_2(t_0,\xi)|^2 \exp\left(2\int_{t_0}^{t} (\phi_{2,\Re} + |b_2|)\ ds\right) \\ \geq |v_2(t_0,\xi)|^2 \exp\left(2\int_{t_0}^{t} (\phi_{2,\Re} - |b_2|)\ ds\right) \end{cases}$$

$$\delta_1 < 1$$
$$\Leftrightarrow \ 4a^2|\xi| > |a'|$$

$$\Rightarrow \begin{cases} v_2(t,\xi) \text{ is defined} \\ C^{-1}|v_1| \leq |v_2| \leq C|v_1| \\ \exp\left(\int_{t_0}^{t} \phi_{2,\Re}\, ds\right) \text{ is bounded} \end{cases}$$

$$\frac{|a'|^3}{|\xi|^2},\ \frac{|a''|}{|\xi|},\ \frac{(a')^2}{|\xi|} \in L^1((t_0,t)) \Longrightarrow \exp\left(\int_{t_0}^{t} |b_2|\, ds\right):\ \text{bdd}$$

$$|a^{(k)}(t)| \le C_k(1+t)^{-k}\ \ (k=1,2),\ \ (1+t_0)|\xi| = N(\gg 1)$$
$$\Rightarrow C^{-1}|v_1(t_0,\xi)| \le |v_1(t,\xi)| \le C|v_1(t_0,\xi)|$$
$$(t \ge t_0)$$



$$Z_H := \{(t,\xi)\ ;\ t \ge t_0\}$$
(Hyperbolic zone)

$$Z_\Psi := \{(t,\xi)\ ;\ t < t_0\}$$
(Pseudo-differential zone)

$$\partial_t V_0 = (\Phi_0 + B_0) V_0, \quad V_0 = \begin{pmatrix} v_0 \\ \overline{v_0} \end{pmatrix} = \begin{pmatrix} \partial_t v + i a_\infty |\xi| v \\ \partial_t v - i a_\infty |\xi| v \end{pmatrix},$$

$$\Phi_0 = \frac{i(a^2 + a_\infty^2)|\xi|}{2a_\infty} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$B_0 = \frac{i(a^2 - a_\infty^2)|\xi|}{2a_\infty} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

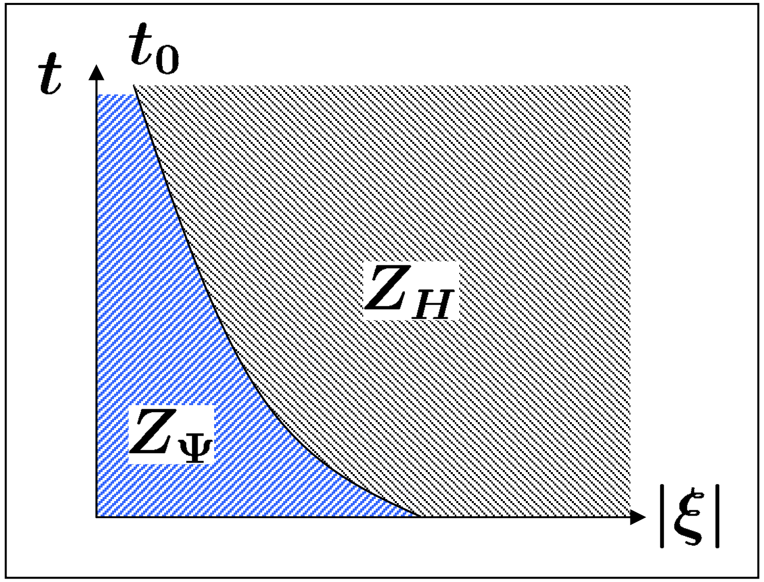
$$\Longrightarrow \quad \partial_t |v_0|^2 = \frac{(a^2 - a_\infty^2)|\xi|}{a_\infty} \Im\{v_0^2\}$$

$$\begin{cases} \leq \frac{|a^2 - a_\infty^2||\xi|}{a_\infty} |v_0|^2 \leq C|\xi||v_0|^2 \\ \leq -\frac{|a^2 - a_\infty^2||\xi|}{a_\infty} |v_0|^2 - C|\xi||v_0|^2 \end{cases}$$

$$\Rightarrow \quad |v_0(t,\xi)|^2$$

$$\begin{cases} \le |v_0(0,\xi)|^2 \exp\left(C|\xi|t_0\right) ds) \le |v_0(0,\xi)|^2 e^{NC} \\ \ge |v_0(0,\xi)|^2 \exp\left(-C|\xi|t_0\right) ds) \ge |v_0(0,\xi)|^2 e^{-CN} \end{cases}$$

$$(0 \le t \le t_0)$$



$$C^{-1}|v_1| \le |v_0| \le C|v_1|$$

$$0 < a_0 \le a(t) \le a_1, \ (1+t)|\xi| < N$$

$$\Rightarrow C^{-1}|v_1(0,\xi)| \le |v_1(t,\xi)| \le C|v_1(0,\xi)|$$

Key of the proof

(i) Division of the phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}^n\}$ into Z_H and Z_Ψ.

(ii) In $Z_H = \{|\xi| \geq N(1+t)^{-1}\}$:

- The transformation $v_1 \to v_2$ is valid. (Diagonalization)
- $a'(t) \in L^1$ does not required; we need the boundedness of
$$(1+t_0)\int_{t_0}^{\infty} (|a'(s)|^2 + |a''(s)|)\, ds.$$

(iii) In $Z_\Psi = \{|\xi| < N(1+t)^{-1}\}$:

- $a \in C^1$ is not necessary; we only use $0 < a_0 \leq a(t) \leq a_1$.

Why can we overcome the infinitely oscillations?

in Z_H

$$|v_2(t,\xi)|^2 \begin{cases} \leq |v_2(t_0,\xi)|^2 \exp\left(2\int_{t_0}^t (\phi_{2,\Re} + |b_2|)\, ds\right) \\ \geq |v_2(t_0,\xi)|^2 \exp\left(2\int_{t_0}^t (\phi_{2,\Re} - |b_2|)\, ds\right) \end{cases}$$

$$\phi_{2,\Re} = \frac{1}{2}\partial_t \left(\log\left(\frac{a}{1-|\delta_1|^2}\right)\right)$$

L^1 in Z_H

$$C_{2-}\frac{a(t)}{a(t_0)}|v_2(t_0,\xi)|^2 \leq |v_2(t,\xi)|^2 \leq C_{2+}\frac{a(t)}{a(t_0)}|v_2(t_0,\xi)|^2$$

$$C_{2\pm} = \frac{1-|\delta_1(t_0)|^2}{1-|\delta_1(t)|^2}\exp\left(\pm 2\int_{t_0}^t |b_2|\, ds\right)$$

We can distinguish the signs of $a'(t)$ in the estimates.

Related results

I. Unbounded $a(t) = \lambda(t)\omega(t)$

$$\begin{cases} \lambda(t)\colon\ C^\infty,\ \ \lambda'(t) \geq 0,\ \ \Lambda(t) = \int_0^t \lambda(s)ds \\ \omega(t)\colon\ C^2,\ \ 0 < \omega_0 \leq \omega(t) \leq \omega_1 \end{cases}$$

$$|\omega^{(k)}(t)| \leq C_k \lambda(t) \left(\frac{\lambda(t)}{\Lambda(t)} \right)^k \ (k = 1, 2) \ \Rightarrow \ (GEC)$$

([Reissig-Yagdjian (2000)])

$$(GEC) \ \Leftrightarrow \ C^{-1} \frac{\lambda(0)}{\lambda(t)} E(0) \leq E(t) \leq C \frac{\lambda(0)}{\lambda(t)} E(0)$$

II. Dissipative wave equation: $\left(\partial_t^2 - \Delta + 2b(t)\partial_t\right) u = 0$

$$\begin{cases} b(t) > 0,\ b(t) < b_1(1+t)^{-1}, \\ b'(t) < 0,\ |b'(t)| \le C(1+t)^{-2} \end{cases}$$

$b_1 < 1,\ u_0 \in H^1,\ u_1 \in L^2$

$$\Rightarrow\ C^{-1} \le (1+t)^{b_1} E(0) \le C$$

([Wirth (2006)])

$$\left(\partial_t^2 - (\lambda(t)\omega(t))^2\Delta\right)u = 0$$

$$\tau = \Lambda(t),\ \ \beta(\tau) = b(t(\tau)),\ \ \mu(\tau) = \lambda(t(\tau))$$

$$b(\tau) = \frac{d}{d\tau}\left\{\log\left(\mu(\tau)\beta(\tau)\right)\right\}$$

$$v(\tau, x) = u(t(\tau), x)$$

$$\left(\partial_t^2 - \Delta + 2b(t)\partial_t\right)u = 0$$

Refined diagonalization procedure

Question:

Can we derive a benefit of $\boldsymbol{C^m}$ $(\boldsymbol{m \geq 3})$ property of $\boldsymbol{a(t)}$?

Review of the diagonalization procedure $\boldsymbol{v_1 \to v_2}$ in $\boldsymbol{Z_H}$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1,$$

$$\Phi_1 = \begin{pmatrix} \frac{a'}{2a} + ia|\xi| & 0 \\ 0 & \frac{a'}{2a} - ia|\xi| \end{pmatrix} = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & -\frac{a'}{2a} \\ -\frac{a'}{2a} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \overline{b_1} \\ b_1 & 0 \end{pmatrix}$$

$$\delta_1 = \frac{ia'}{4a^2|\xi|} = \frac{-ib_1}{2\phi_{1,\Im}}$$

$$\partial_t V_1 = (\Phi_1 + B_1)V_1$$

$$V_2 = M_1^{-1} V_1, \quad M_1 = \begin{pmatrix} 1 & \overline{\delta_1} \\ \delta_1 & 1 \end{pmatrix}, \quad \delta_1 = \frac{-ib_1}{2\phi_{1,\Im}}$$

$$\partial_t V_2 = (\Phi_2 + B_2)V_2$$

$$\Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & \overline{b_2} \\ b_2 & 0 \end{pmatrix},$$

$$\phi_{2,\Re} = \frac{1}{2}\partial_t \left(\log \left(\frac{a}{1 - |\delta_1|^2} \right) \right), \quad \phi_{2,\Im} = a|\xi| - \frac{2|\delta_1|^2}{1 - |\delta_1|^2}$$

$$|b_1| \le C(1+t)^{-1}$$

$$|\delta_1| \le C(1+t)^{-1}|\xi|^{-1}, \ |\delta_1'| \le C(1+t)^{-2}|\xi|^{-1}$$

$$|b_2| \le C(1+t)^{-2}|\xi|^{-1} \qquad \left(b_2 = \frac{b_1|\delta_1|^2 - \delta_1'}{1 - |\delta_1|^2} \right)$$

$$\partial_t V_j = (\Phi_j + B_j) V_j$$

$$V_j = \begin{pmatrix} v_j \\ \overline{v_j} \end{pmatrix}, \Phi_j = \begin{pmatrix} \phi_j & 0 \\ 0 & \overline{\phi_j} \end{pmatrix}, \quad B_j = \begin{pmatrix} 0 & \overline{b_j} \\ b_j & 0 \end{pmatrix}$$

$$V_{j+1} = M_j^{-1} V_j, \quad M_j = \begin{pmatrix} 1 & \overline{\delta_j} \\ \delta_j & 1 \end{pmatrix}, \quad \delta_j = \frac{-i b_j}{2\phi_{j,\Im}}$$

$$\partial_t V_{j+1} = (\Phi_{j+1} + B_{j+1}) V_{j+1}$$

Lemma. $a(t) \in C^m, \ V_m = M_{m-1}^{-1} \cdots M_1^{-1} V_1,$

$|\delta_j| < 1 \ (j = 1, \cdots, m-1) \ \Rightarrow \ \partial_t V_m = (\Phi_m + B_m) V_m$

$$V_1 = {}^t(v_1, \overline{v_1}), \ v_1 = \partial_t v + ia|\xi| v$$

$$\Phi_j = \begin{pmatrix} \phi_j & 0 \\ 0 & \overline{\phi_j} \end{pmatrix}, \ B_j = \begin{pmatrix} 0 & \overline{b_j} \\ b_j & 0 \end{pmatrix}$$

$$\begin{cases} \phi_{j+1,\Re} = \dfrac{1}{2}\partial_t \left(\log \left(\dfrac{a}{\prod_{k=1}^{j} (1 - |\delta_k|^2)} \right) \right) \\ \phi_{j+1,\Im} = a|\xi| + \displaystyle\sum_{k=1}^{j} \dfrac{-2|\delta_k|^2 \phi_{k,\Im} + \Im\{\delta_k' \overline{\delta_k}\}}{1 - |\delta_k|^2} \\ b_{j+1} = \dfrac{b_j |\delta_j|^2 - \delta_j'}{1 - |\delta_j|^2} \qquad (j = 0, \cdots, m-1) \end{cases}$$

$$|v_m(t,\xi)|^2 \begin{cases} \leq |v_m(t_0,\xi)|^2 \exp\left(2 \int_{t_0}^{t} (\phi_{m,\Re} + |b_m|) \ ds\right) \\ \geq |v_m(t_0,\xi)|^2 \exp\left(2 \int_{t_0}^{t} (\phi_{m,\Re} - |b_m|) \ ds\right) \end{cases}$$

(GEC) for very fast oscillating coefficient

$$|a'(t)| \le C(1+t)^{-\beta}$$

$\beta > 1 \;\Rightarrow$ (GEC) $\cdots$ ***very slow oscillation*** *(trivial case)*

$\beta = 1 \;\Rightarrow$ (GEC) $\cdots$ ***slow oscillation*** *(critical case)*

$\beta < 1 \;\Rightarrow$ no (GEC) $\cdots$ ***very fast oscillation***

Contribution of C^2 property of $a(t)$:

$$|a''(t)| \le C(1+t)^{-2}$$

The refined diagonalization with $\boldsymbol{C^m}$ property of $\boldsymbol{a(t)}$:

$$|a^{(k)}(t)| \leq C(1+t)^{-k\beta} \quad (k = 1, \cdots, m)$$

can conclude (GEC) for very fast oscillation $\beta < 1$!

if $a(t)$ satisfies the stabilization property.

Contradiction to the example of no (GEC):

$$a(t) = 2 + \cos\left((1+t)^p\right) \quad (p > 0)$$

does not satisfy the stabilization property

$$\int_0^t |a(s) - a_\infty| ds \leq C(1+t)^\alpha \quad (0 \leq \alpha < 1)$$

(stabilization property)

$$\mathcal{Z}_\Psi = \{t < \tau_0\}, \quad \mathcal{Z}_H = \{t \geq \tau_0\}, \quad (1+\tau_0)^\alpha |\xi| = N$$

$\underline{m = 2}$

in $\mathcal{Z}_H$

$$\partial_t V_2 = (\Phi_2 + B_2) V_2, \quad B_2 = \begin{pmatrix} 0 & \overline{b_2} \\ b_2 & 0 \end{pmatrix}, \quad \delta_1 = i \frac{a'}{4a^2|\xi|}$$

$$b_2 = \frac{1}{1-\delta_1}\left(-\frac{(a')^3}{32a^5|\xi|^2} - i\left(\frac{a''}{4a^2|\xi|} - \frac{(a')^2}{2a^3|\xi|} \right)\right)$$

$$|\delta_1| \leq C(1+t)^{-\beta}|\xi|^{-1} = CN^{-1}(1+t)^{-\beta}(1+\tau_0)^\alpha \ll 1$$

$$(\alpha \leq \beta, \ N \gg 1 \ )$$

$$\int_{\tau_0}^t |b_2| ds \leq C \int_{\tau_0}^t (1+t)^{-2\beta}|\xi|^{-1} ds$$

$$\leq C(1+\tau_0)^{\alpha-2\beta+1} \leq C \quad \left(\frac{\alpha+1}{2} = \beta (< 1) \right)$$

in $\mathcal{Z}_\Psi$

$$\partial_t V_0 = (\Phi_0 + B_0) V_0, \quad V_0 = \begin{pmatrix} v_0 \\ \overline{v_0} \end{pmatrix} = \begin{pmatrix} \partial_t v + i a_\infty |\xi| v \\ \partial_t v - i a_\infty |\xi| v \end{pmatrix},$$

$$\Phi_0 = \frac{i(a^2 + a_\infty^2)|\xi|}{2a_\infty} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_0 = \frac{i(a^2 - a_\infty^2)|\xi|}{2a_\infty} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$\Longrightarrow$ $$\partial_t |v_0|^2 \le C|a - a_\infty||\xi||v_0|^2$$

$\Longrightarrow$ $$|v_0(t)|^2 \le \exp\left(C|\xi| \int_0^{\tau_0} |a(s) - a_\infty| ds \right) |v_0(0)|^2$$

$$\le \exp\left(C|\xi|(1 + \tau_0)^\alpha \right) |v_0(0)|^2$$

$$= e^{CN} |v_0(0)|^2$$

$$m = 2,\ \beta = \frac{\alpha + 1}{2} \ \Rightarrow\ (GEC)$$

<u>$m \geq 3$</u>

$$|a^{(k)}(t)| \leq C_k(1+t)^{-k\beta}\ (1 \leq k \leq m)$$

Theorem. ([H. (2007)])

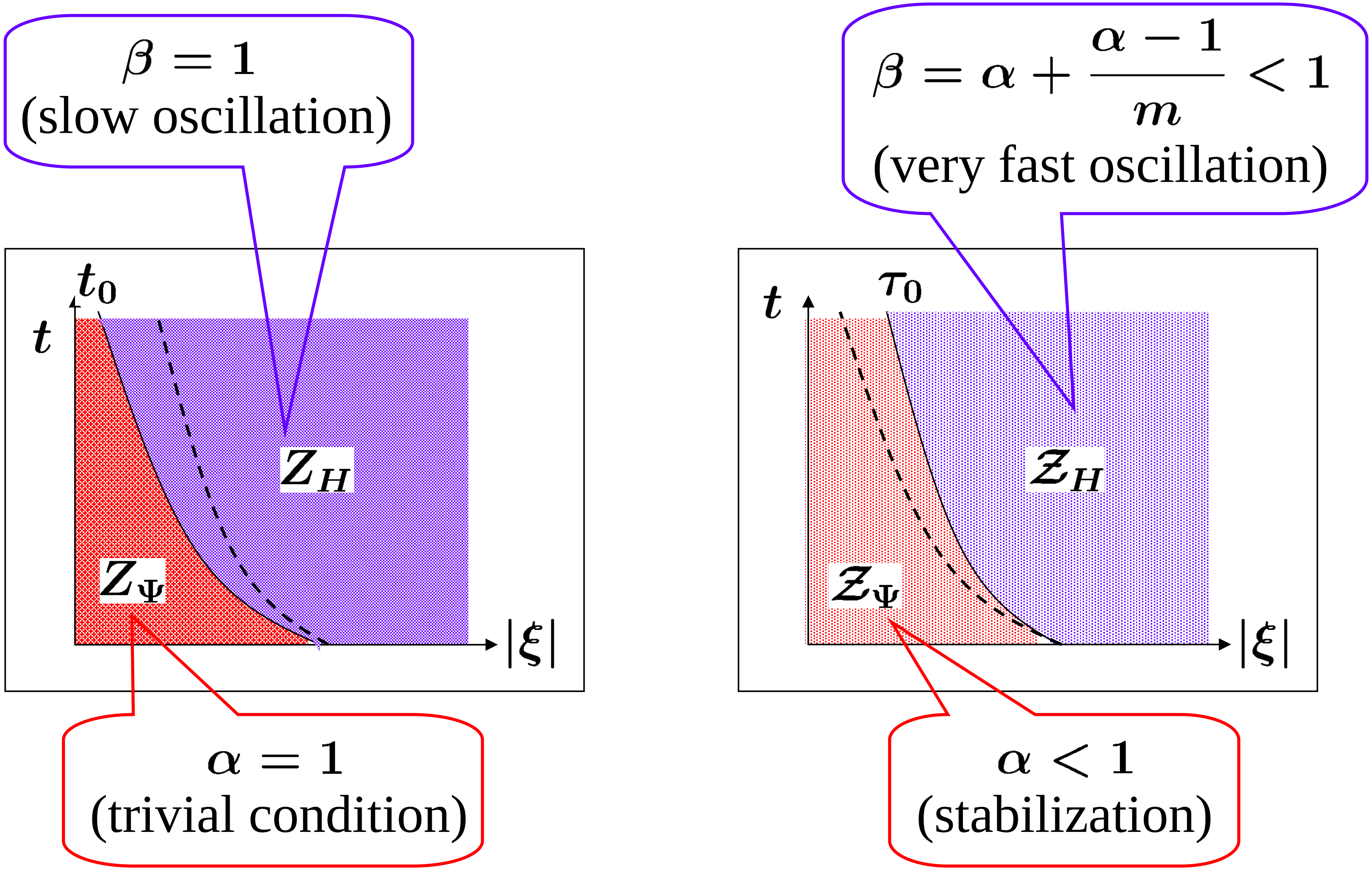
$$\beta = \alpha + \frac{1 - \alpha}{m} \ \Rightarrow\ (GEC)$$

Corollary.

$$|a^{(k)}(t)| \leq C_k(1+t)^{-k\beta}\ (k \in \mathbb{Z})$$

$$\beta > \alpha \ \Rightarrow\ (GEC)$$

$$\int_0^t |a(s) - a_\infty| ds \le C(1+t)^\alpha, \ |a'(t)| \le C(1+t)^{-\beta}$$



Related results (C^m property of the coefficients)

I. Unbounded propagation speed: [H. -Wirth, preprint]

II. Dissipative wave equations: [H. -Wirth, 2008]

III. Applications to Kirchhoff type equations:

[Manfrin, 2005], [H., 2007]

Consideration near the critical case

$$|a^{(k)}(t)| \le C_k (1+t)^{-k\beta} \ (k \in \mathbb{Z}),\ \beta > \alpha \ \Rightarrow \ (GEC)$$

$$a(t) \in C^\infty \ \Rightarrow \ a(t) \in \gamma^s \ \text{(Gevrey class)}$$

Theorem. ([H.])

$$|a^{(k)}(t)| \le k!^s \left(C(1+t)^\alpha (\log(e+t))^\delta \right)^{-k} \ (k \in \mathbb{Z})$$

$$\delta \ge s \ \Rightarrow \ (GEC)$$

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