On wave equations with time dependent coefficients of the Gevrey class

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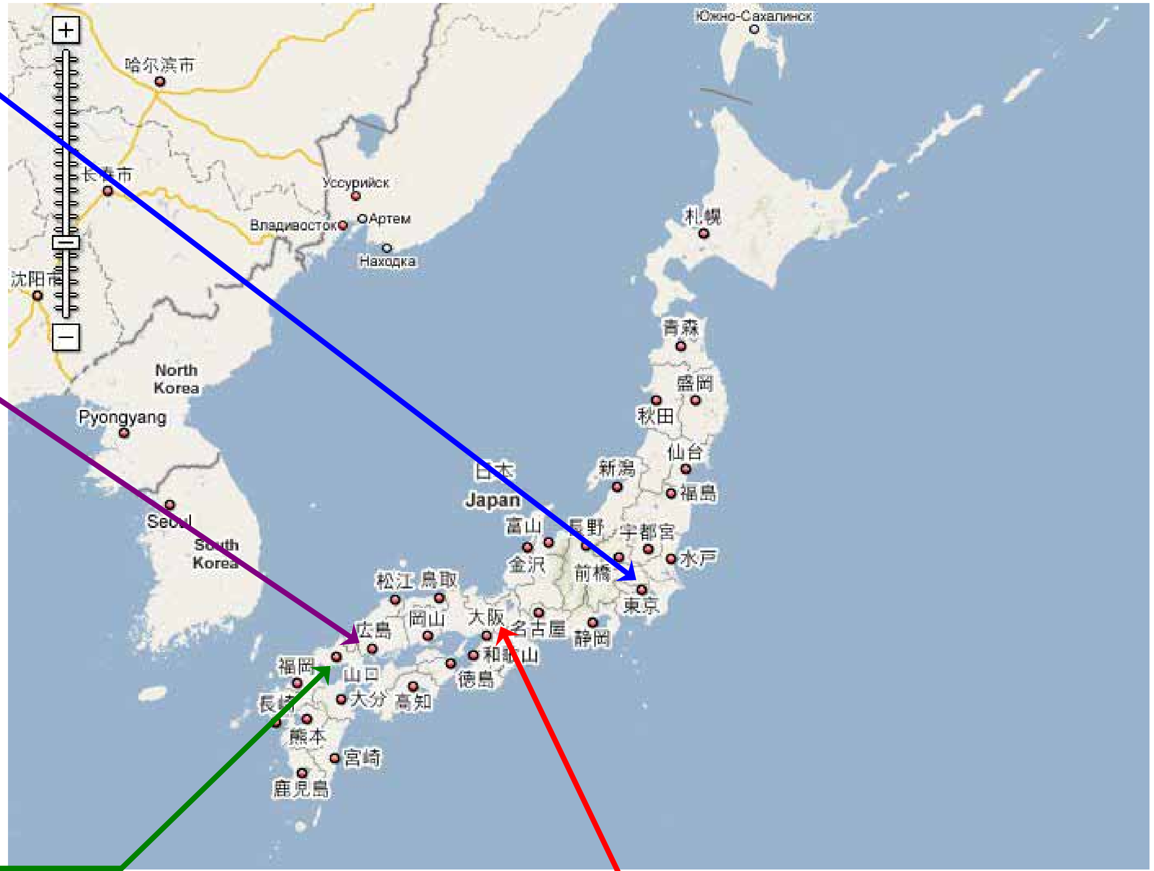

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Introduction

Kirchhoff equation (vibration of elastic string):

$$\text{(K)} \qquad \left(\partial_t^2 - \left(1 + \int_I |\partial_x u(t,x)|^2 dx\right) \partial_x^2\right) u(t,x) = 0$$

$$\Longrightarrow \quad \left(\partial_t^2 - a(t)^2 \partial_x^2\right) u(t,x) = 0$$

$$E_s(t) = \frac{1}{2}\left(a(t)^2 \|\partial u(t,\cdot)\|_s^2 + \|\partial_t u(t,\cdot)\|_s^2\right)$$

$$\begin{aligned} E_s'(t) &= a'(t)a(t)\|\partial u(t,\cdot)\|_s^2 \\ &= 2a(t)\Re(\partial\partial_t u(t,\cdot), \partial u(t,\cdot))_s \|\partial u(t,\cdot)\|_s^2 \\ &\leq 2a(t)\|\partial_t u(t,\cdot)\|_{\frac{1}{2}} \|\partial u(t,\cdot)\|_{\frac{1}{2}} \|\partial u(t,\cdot)\|_s^2 \\ &\leq C E_s(t)^2 \end{aligned}$$

$E_s(t) \leq (E_s(0) - Ct)^{-1}$ $\cdots$ time local estimate!

Consider the following Cauchy problem:

$$(C) \begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, & (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n. \end{cases}$$

(A0) $$0 < a_0 \le a(t) \le a_1$$

[Colombini - De Giorgi - Spagnolo (1979)]

$a \in C^\alpha([0,\infty)) \Rightarrow$ (C) *is* γ^s *well-posed with* $s < \frac{1}{1-\alpha}$

$f(x) \in \gamma^s$ (Gevrey class of order s)

$$\Leftrightarrow \; |\hat{f}(\xi)| \le C \exp\left(-\rho|\xi|^{\frac{1}{s}}\right) \;\; (s > 1,\ \exists C > 0,\ \exists \rho > 0)$$

$$\forall s > \frac{1}{\alpha-1},\ |(\hat{u}(0,\xi), \hat{u}_t(0,\xi))| \exp\left(\rho|\xi|^{\frac{1}{\tilde{s}}}\right) < \infty,\ \exists T > 0$$

$$\lim_{t\to T-0} |(\hat{u}(t,\xi), \hat{u}_t(t,\xi))| \exp\left(-\rho|\xi|^{\frac{1}{\tilde{s}}}\right) = \infty$$

Well-posedness in γ^s for $a \in C^\alpha$

Regularity of $a(t)$

Singular ⟷ **Regular**

α	0	α	1	$\cdots$
$\frac{1}{s}$	1	$1-\alpha$	0 (L^2 well-posed)	

Big ⟷ **Small**

Regularity loss

$\underline{a(t) \in C^1([0,\infty))}$

$$E(t) = \frac{1}{2}\left(a(t)^2 \, \|\nabla u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \right)$$

$$a(t) \equiv const. \;\Rightarrow\; E(t) \equiv E(0)$$ *(Energy Conservation)*

$$a(t) \not\equiv const. \;\Rightarrow\; E(t) \not\equiv E(0)$$

$$a'(t) > 0 \;\Rightarrow\; E'(t) \geq 0, \;\; a'(t) < 0 \;\Rightarrow\; E'(t) \leq 0$$

$$E'(t) = a'(t)\, a(t)\, \|\nabla u(t,\cdot)\|^2 \begin{cases} \leq \frac{2|a'(t)|}{a(t)} E(t) \\ \geq -\frac{2|a'(t)|}{a(t)} E(t) \end{cases}$$

$$\Longrightarrow\; E(t) \begin{cases} \leq \exp\left(C \int_0^t |a'(\tau)|\, d\tau \right) E(0) \\ \geq \exp\left(-C \int_0^t |a'(\tau)|\, d\tau \right) E(0) \end{cases}$$

$\underline{a(t) \in C^1([0,\infty))}$

$$|a'(t)| \leq C \implies E(t) \lesseqgtr E(0) \exp\left(\pm\rho(1+t)\right)$$

$$a'(t) \in L^1(\mathbb{R}^+) \implies E(t) \lesseqgtr C^{\pm 1} E(0) \quad \cdots (GEC)$$

(GEC) = *Generalized Energy Conservation*

$$|a'(t)| \leq C(1+t)^{-\beta}, \ (0 < \beta \leq 1)$$

$$\implies E(t) \begin{cases} \lesseqgtr E(0) \exp\left(\pm\rho(1+t)^{-\beta+1}\right) & (\beta < 1) \\ \lesseqgtr E(0)(1+t)^{\pm M} & (\beta = 1) \end{cases}$$

Main purpose: We realize a benefit of C^m property of $a(t)$ on the estimate of $E(t)$.

C^2 property without a stabilization of the coefficients

Example. $-1 \leq \mu(\tau) \leq 1$, $\mu \in C^1$ and 1-periodic

$$a(t) = 2 + \mu(\log(e+t))$$

$\Longrightarrow$ $|a'(t)| \leq C(1+t)^{-1}$

C^1-property; $\mu \in C^1 \implies E(t) \lesseqgtr E(0)(1+t)^{\pm M}$

C^2-property; $\mu \in C^2 \implies E(t) \lesseqgtr C^{\pm 1} E(0) \quad \cdots (GEC)$

Theorem. ([Reissig-Smith (2005)])

$$|a^{(k)}(t)| \leq C(1+t)^{-k} \quad (k=1,2) \implies (GEC)$$

Remark.

(i) $|a'(t)| \le C(1+t)^{-1}$ is sharp for (GEC).

(ii) C^2 property of $a(t)$ is required.

(iii) Necessity of the condition to $a''(t)$ is an open problem.

Remark.

$$a(t) = 2 + \mu(\omega(t)),\ \ \omega'(t) \ge 0$$

$$\Longrightarrow \quad \int_0^T |a'(t)|dt \simeq \int_0^T |\omega'(t)|dt = \omega(T):$$

number of oscillations on $[0, T]$

$$a'(t) \in L^1((0,\infty)) \ \Leftrightarrow \ \text{finite numbers of oscillations}$$

We have (GEC) for an infinitely oscillating coefficient!

$$E'(t) \begin{cases} \geq 0 \quad \text{for} \quad a'(t) > 0 \quad \Rightarrow \quad (E(t) \nearrow) \\ \leq 0 \quad \text{for} \quad a'(t) < 0 \quad \Rightarrow \quad (E(t) \searrow) \end{cases}$$

some cancelations of oscillating energy

C^2-property

$$C^{-1}E(0) \leq E(t) \leq CE(0)$$

Remark. We can separate the oscillation of the coefficient by a precise representation of the microenergy in high frequency on C^2-property:

$$C_{2-}\frac{a(t)}{a(t_0)}\mathcal{E}(t_0,\xi) \leq \mathcal{E}(t,\xi) \leq C_{2+}\frac{a(t)}{a(t_0)}\mathcal{E}(t_0,\xi)$$

$$|a'(t)| \leq C(1+t)^{-\beta}$$

$\beta > 1 \;\Rightarrow$ (GEC) $\cdots$ ***very slow oscillation*** *(trivial case)*

$\beta = 1 \;\Rightarrow$ (GEC) $\cdots$ ***slow oscillation*** *(critical case)*

$\beta < 1 \;\Rightarrow$ no (GEC) $\cdots$ ***very fast oscillation***

Contribution of C^2 property of $a(t)$:

$$|a''(t)| \leq C(1+t)^{-2}$$

Question. Can (GEC) hold under additional assumptions to higher order derivatives for *very fast oscillating* coefficients?

C^m property with a stabilization of the coefficients

(GEC) does not hold for the very fast oscillating coefficients:

$$a(t) = 2 + \cos\left((1+t)^p\right) \quad (p > 0)$$

However, (GEC) can be valid under the assumption to the coefficient, which is called the *stabilization property*, though very fast oscillation.

$$\int_0^t |a(s) - a_\infty| ds \le C(1+t)^\alpha \quad (0 \le \alpha < 1)$$

(stabilization property)

Theorem. ([H. (2007)] cf. [Cicognani - H. (2008)]) .

$$|a^{(k)}(t)| \le C_k(1+t)^{-k\beta} \ (1 \le k \le m,\ m \ge 2),\ \alpha \le \beta$$

$$\Longrightarrow \quad E(t) \lesseqgtr E(0)\exp\left(\pm\rho(1+t)^{\sigma}\right)$$

$$\sigma = \max\left\{0, \frac{1+\alpha(m-1)}{m} - \beta\right\}$$

Corollary. $|a^{(k)}(t)| \le C_k(1+t)^{-k\beta} \ (k \in \mathbb{Z}),\ \alpha \le \beta$

$$\Longrightarrow \quad E(t) \lesseqgtr E(0)\exp\left(\pm\rho(1+t)^{\sigma}\right)$$

$$\sigma = \begin{cases} 0 & (\alpha < \beta) \\ \forall\varepsilon > 0 & (\alpha = \beta) \end{cases}$$ **(critical case!)**

Remark. $\beta < \alpha \Rightarrow$ exists a counterexample of non-(GEC)

On the estimates $\boldsymbol{E(t) \lesseqgtr E(0)\exp\left(\pm\rho(1+t)^{\sigma}\right)}$

$$\int_0^t |a(s) - a_\infty|\,ds \lesssim (1+t)^{\alpha},\ \ |a'(t)| \le C(1+t)^{-\beta}$$
$$(0 \le \alpha \le \beta < 1)$$

Regularity of $\boldsymbol{a}$

Singular ⟷ *Regular*

m	1	2	$\dots$	m	$\dots$	∞
σ	$1-\beta$	$\frac{\alpha+1}{2}-\beta$	$\dots$	$\frac{1+\alpha(m-1)}{m}-\beta$	$\dots$	$\alpha-\beta=0$

Unstable ⟷ *Stable*

Behavior of $\boldsymbol{E(t)}$

The refined diagonalization with $\boldsymbol{C^m}$ property of $\boldsymbol{a(t)}$:

$$|a^{(k)}(t)| \le C(1+t)^{-k\beta} \quad (k = 1, \cdots, m)$$

can conclude (GEC) for very fast oscillation $\beta < 1$!

if $a(t)$ satisfies the stabilization property.

Contradiction to the example of no (GEC):

$$a(t) = 2 + \cos\left((1+t)^p\right) \quad (p > 0)$$

does not satisfy the stabilization property

$$\int_0^t |a(s) - a_\infty| ds \le C(1+t)^\alpha \quad (0 \le \alpha < 1)$$

(stabilization property)

Critical case for the Gevrey coefficients

$$|a^{(k)}(t)| \leq C_k (1+t)^{-k\beta} \quad (k \in \mathbb{Z},\ \alpha < \beta)$$

$$\Downarrow \ \beta \to \alpha$$

(G) $$|a^{(k)}(t)| \leq C k!^{\nu} \left((1+t)^{\alpha} (\log(e+t))^{\delta} \right)^{-k} \quad (k \in \mathbb{Z})$$

> **Theorem.** $\int_0^t |a(s) - a_\infty| ds$, (G), $\nu > 1 \Rightarrow$
>
> $$E(t) \lesseqgtr E(0) \exp\left(\pm\rho \left(\log(e+t)\right)^{\sigma}\right), \quad \sigma = \max\{0, \nu - \delta\}$$

Regularity of a

Singular ⟷ *Regular*

ν	∞	$\cdots$	ν	$\cdots$	1	
σ	∞	$\cdots$	$\nu - \delta$	$\cdots$	$1 - \delta$	$(\nu \geq \delta)$

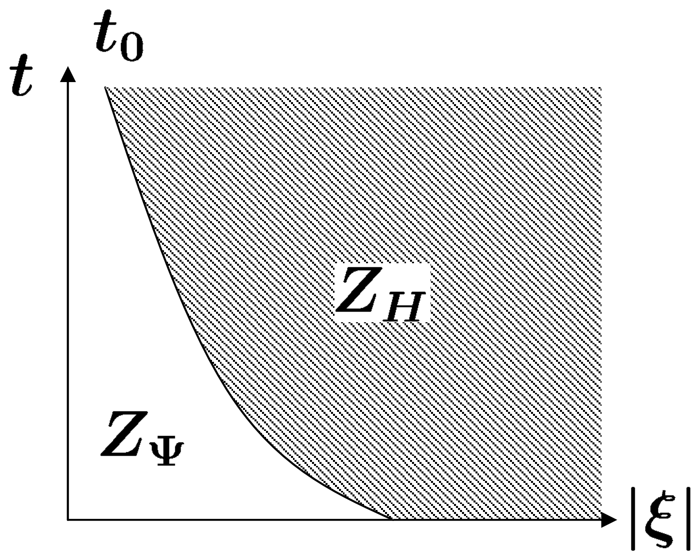
Unstable ⟷ *Stable*

Behavior of $E(t)$

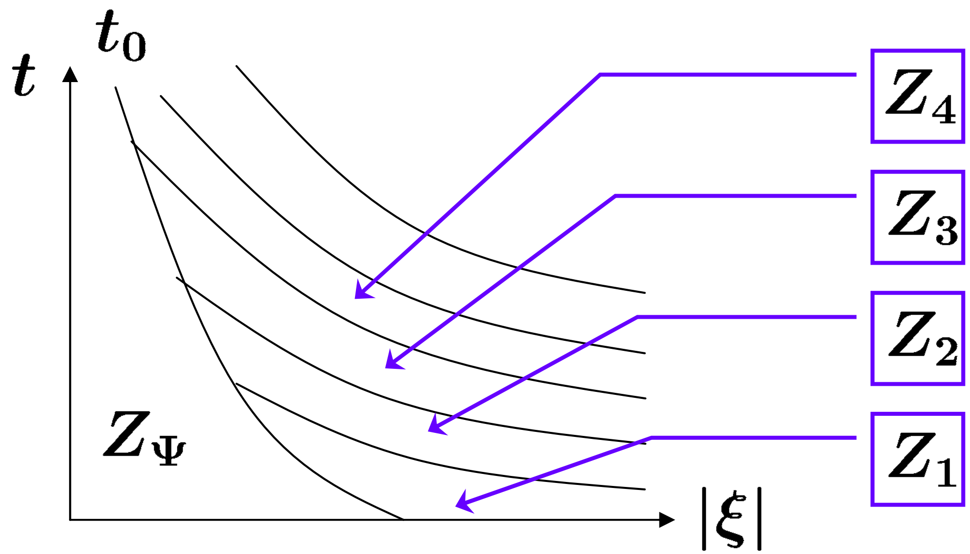
Key of the proof

- Refined diagonalization
- Division of infinitely many zones
- Algebra of the Gevrey functions

$a \in C^m$



$a \in \gamma^\nu$



$$(1+t_0)^\alpha|\xi| = N$$

- Division of infinitely many zones

$$Z_k = \{(t, \xi) \, ; \, t_{k-1} \le t \le t_k\} \, ,$$

$$(1 + t_k)^{\alpha} \, (\log(e + t_k))^{\delta} \, |\xi| = (k + 1)^{\nu}$$

- Algebra of the Gevrey functions

$$\sum_{j=0}^{n} \binom{n}{k} \left(\frac{k!(n-k)!}{n!} \right)^{\nu} \le C \quad \Leftrightarrow \quad \nu > 1$$

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[3] F. Hirosawa, *Math. Ann.* **339** (2007).

[4] M. Cicognani - F. Hirosawa, *Math. Scand.* **102** (2008).