Wave equations with time dependent coefficients

Fumihiko Hirosawa

Yamaguchi University, JAPAN

1. Introduction

$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n, \end{cases} \tag{1}$$

$$E(t) = \frac{1}{2}\left(a(t)^2 \,\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\partial_t u(t,\cdot)\|_{L^2}^2 \right)$$

$$0 < a_0 \le a(t) \le a_1,\; a(t) \in C^1([0,\infty))$$

$a(t) \equiv a_0 \; (const.) \;\Rightarrow\; E(t) \equiv E(0)$ (Energy conservation)

$$a(t) \not\equiv a_0 \;\Rightarrow\; E'(t) = a'a\|\nabla u\|^2 \not\equiv 0 \;\Rightarrow\; E(t) \not\equiv E(0)$$

2. Trivial estimates by C^1 property of the coefficients

$$E'(t) = a'(t)a(t)\|\nabla u(t,\cdot)\|^2$$

$$\Longrightarrow \quad -\frac{2|a'(t)|}{a(t)}E(t) \le E'(t) \le \frac{2|a'(t)|}{a(t)}E(t)$$

$$\Longrightarrow \quad E(t) \begin{cases} \ge \exp\left(-\frac{2}{a_0}\int_0^t |a'(s)|ds\right)E(0) \\ \le \exp\left(\frac{2}{a_0}\int_0^t |a'(s)|ds\right)E(0) \end{cases}$$

$$|a'(t)| \le C_1(1+t)^{-\beta} \ (\beta \ge 0)$$

Example: $a(t) = 2 + \cos\left((1+t)^{-\beta+1}\right) \quad (\beta < 1)$

$$e^{-\eta(t)}E(0) \leq E(t) \leq e^{\eta(t)}E(0) \quad (t \to \infty) \quad (2)$$

$$(\eta(t) > 0, \quad \eta'(t) \geq 0)$$

Theorem 2.1. $|a'(t)| \lesssim (1+t)^{-\beta} \Rightarrow$ *(2) holds for*

$$\eta(t) \simeq \begin{cases} t^{-\beta+1} & (0 \leq \beta < 1) \\ \log t & (\beta = 1) \\ 1 & (\beta > 1) \end{cases}$$

Question. Can the estimate (2) be improved or not under some additional assumptions to $a(t)$?

$\underline{\eta(t) \simeq 1}$

$$C^{-1}E(0) \le E(t) \le CE(0) \;\Leftrightarrow\; E(t) \simeq E(0)$$

Generalized energy conservation (= GEC)

Theorem 2.1. $|a'(t)| \lesssim (1+t)^{-\beta},\ \beta > 1 \Rightarrow \text{(GEC)}$

<u>Main purpose</u>

Take β smaller as far as it is possible under some additional assumptions to higher order derivatives of the coefficient.

3. C^2 property in the critical case

Theorem 3.1. ([Reissig-Smith (2005)])

$$a(t) \in C^2,\ |a^{(k)}(t)| \le C_k(1+t)^{-k}\ (k=1,2)$$

$$\Rightarrow \text{(GEC)}: (E(t) \simeq E(0)) \qquad (\beta = 1)$$

Example: $a(t) = 2 + \cos(\log(1+t))$

Theorem 3.2. (GEC) *does not hold for*

$$a(t) = 2 + \cos\left(\log(1+t)^{1+\varepsilon}\right) \quad (\forall \varepsilon > 0).$$

$$|a'(t)| \le C(1+t)^{-1}(\log(e+t))^{\varepsilon}$$

$$a(t) = 2 + \cos((1+t)^{\varepsilon}) \quad (\forall \varepsilon > 0).$$

$$|a'(t)| \le C(1+t)^{-1+\varepsilon}$$

4. C^m property in the super-critical case

Theorem 4.1. ([H. (2007)]) $m \geq 2$,

$$a(t) \in C^m,\ |a^{(k)}(t)| \leq C_k(1+t)^{-k\beta} \ (k = 1, \cdots, m)$$

$$\exists \alpha \in [0,1),\ \int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \ (t \to \infty)$$

$$a_\infty = \lim_{t\to\infty} \frac{\int_0^t a(s)ds}{t}$$

$$\Rightarrow \quad \text{(GEC)} \ \textit{hold for } \beta \geq \beta_m := \alpha + \frac{1-\alpha}{m}.$$

$$\lim_{m\to\infty} \beta_m = \alpha \Rightarrow \ \beta = \alpha \text{ is the limit case as } m \to \infty$$

Corollary. $(m = \infty)$ $\int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \ (t \to \infty)$

$$a(t) \in C^\infty,\ |a^{(k)}(t)| \leq C_k(1+t)^{-k\beta} \ (\forall k \in \mathbb{N})$$

$$\Rightarrow \quad \text{(GEC)} \ \textit{hold for } \forall \beta > \alpha.$$

Summary of the conditions for (GEC)

$$|a'(t)| \leq C(1+t)^{-\beta}$$

$$\left(\text{ with } |a^{(k)}(t)| \leq C_k(1+t)^{-k\beta} \ (k=2,\cdots,m)\right)$$

$$\int_0^t |a(s)-a_\infty|ds = O(t^\alpha) \ \ (t\to\infty)$$

Regularity of a

Singular ⟷ **Regular**

m	1	2	$\cdots$	m	$\cdots$	∞
β	1	$\alpha+\frac{1-\alpha}{2}$	$\cdots$	$\alpha+\frac{1-\alpha}{m}$	$\cdots$	α

Fast ⟷ **Slow**

Decay order of $|a'(t)|$

$(0 \leq \alpha < 1)$

4. Idea of the proof of Theorem 3.1

$$\left(\partial_t^2 - a(t)^2\Delta\right) u(t,x) = 0$$

⇓ Fourier transformation $(\hat{u}(t,\xi) = v(t,\xi))$

$$\left(\partial_t^2 + a(t)^2|\xi|^2\right) v(t,\xi) = 0$$

$$C^{-1}E(0) \le E(t) \le CE(0)$$

⇕

$$C^{-1}\mathcal{E}(0,\xi) \le \mathcal{E}(t,\xi) \le C\mathcal{E}(0,\xi) \quad (\xi \in \mathbb{R}^n)$$

$$\mathcal{E}(t,\xi) = \frac{1}{2}\left(a(t)^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right)$$

$$(\partial_t^2 + a(t)^2|\xi|^2)\, v(t,\xi) = 0$$

⇩

$$\partial_t V_1 = (\Phi_1 + B_1)V_1, \quad V_1 = \begin{pmatrix} v_1 \\ \overline{v_1} \end{pmatrix} = \begin{pmatrix} \partial_t v + ia|\xi|v \\ \partial_t v - ia|\xi|v \end{pmatrix}$$

$$\Phi_1 = \begin{pmatrix} \frac{a'}{2a} + ia|\xi| & 0 \\ 0 & \frac{a'}{2a} - ia|\xi| \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -\frac{a'}{2a} \\ -\frac{a'}{2a} & 0 \end{pmatrix}$$

Remark. $\partial_t V = (\Phi + B)V$, $\phi_{\Re} = \Re\{\phi\}$, $\phi_{\Im} = \Im\{\phi\}$,

$$\left|\int_{t_0}^{t} \phi_{\Re}\, ds\right| \lesssim 1, \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & \overline{\phi} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \overline{b} \\ b & 0 \end{pmatrix}$$

$$\Rightarrow \ |V(t,\xi)| \begin{cases} \lesssim |V(t_0,\xi)| \exp\left(C \int_{t_0}^{t} |b(s,\xi)|\, ds\right) \\ \gtrsim |V(t_0,\xi)| \exp\left(-C \int_{t_0}^{t} |b(s,\xi)|\, ds\right) \end{cases}$$

$$\partial_t V_1 = (\Phi_1 + B_1) V_1$$

$$\Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & \overline{b_1} \\ b_1 & 0 \end{pmatrix}$$

Diagonalization

$$V_2 = M_1^{-1} V_1, \quad M_1 = \begin{pmatrix} 1 & \overline{\delta_1} \\ \delta_1 & 1 \end{pmatrix}, \quad \delta_1 = \frac{-ib_1}{2\phi_{1,\Im}}$$

$$|\delta_1| = \frac{|\frac{a'}{2a}|}{2a|\xi|} \le \frac{C_1(1+t)^{\alpha-\beta}}{4a_0^2 N} \le \frac{C_1}{4a_0^2 N} \le \frac{1}{2}$$

$$(t,\xi) \in Z_H := \{(t,\xi)\,;\ (1+t)^\alpha|\xi| \ge N\}$$

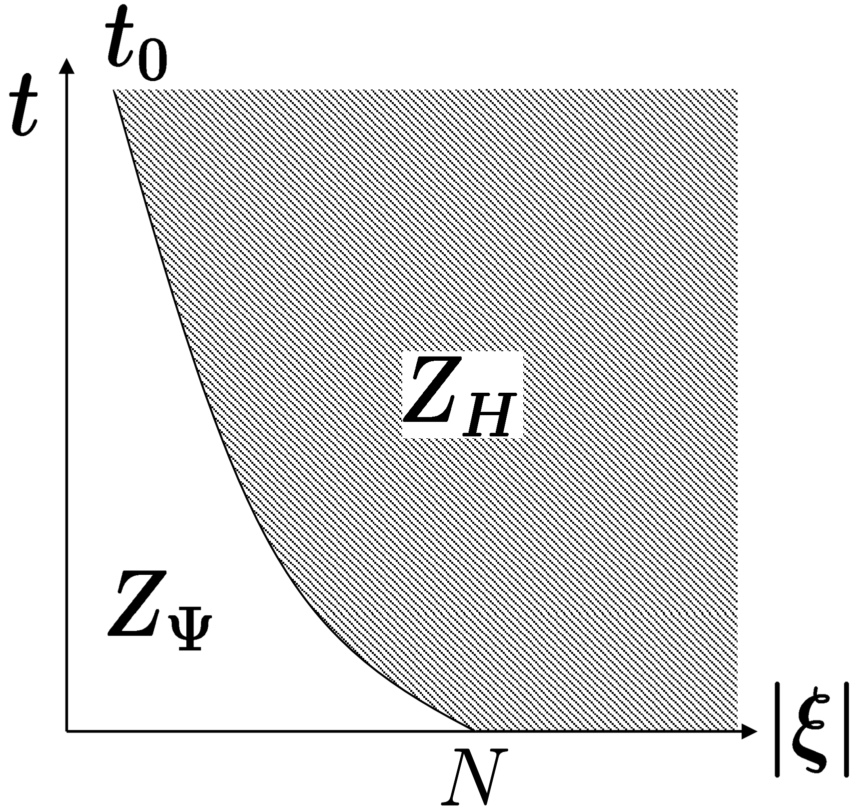
$$(0 \le \alpha \le \beta, \quad N \gg 1)$$

$$\partial_t V_2 = (\Phi_2 + B_2) V_2$$

$$\Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & \overline{b_2} \\ b_2 & 0 \end{pmatrix}$$

$$\phi_{2,\Re} = \frac{1}{2}\partial_t \left(\log \left(\frac{a}{1-|\delta_1|^2} \right) \right), \quad \phi_{2,\Im} = a|\xi| - \frac{2|\delta_1|^2}{1-|\delta_1|^2}$$

$$b_2 = \frac{b_1|\delta_1|^2 - \delta_1'}{1-|\delta_1|^2}$$



$$(1+t_0)^\alpha|\xi| = N$$
$$(N \gg 1)$$

$$Z_H := \{(t,\xi)\,;\ (1+t)^\alpha|\xi| \geq N\}$$

$$Z_\Psi := \{(t,\xi)\,;\ (1+t)^\alpha|\xi| < N\}$$

$$\partial_t V_j = (\Phi_j + B_j) V_j$$

j th step of Diagonalization

$$V_j = \begin{pmatrix} v_j \\ \overline{v_j} \end{pmatrix}, \Phi_j = \begin{pmatrix} \phi_j & 0 \\ 0 & \overline{\phi_j} \end{pmatrix}, \; B_j = \begin{pmatrix} 0 & \overline{b_j} \\ b_j & 0 \end{pmatrix}$$

$$V_{j+1} = M_j^{-1} V_j, \quad M_j = \begin{pmatrix} 1 & \overline{\delta_j} \\ \delta_j & 1 \end{pmatrix}, \quad \delta_j = \frac{-ib_j}{2\phi_{j,\Im}}$$

$$\partial_t V_{j+1} = (\Phi_{j+1} + B_{j+1}) V_{j+1}$$

M_j is invertible only in Z_H!

$$\begin{cases} \phi_{j+1,\Re} = \dfrac{1}{2}\partial_t \left(\log \left(\dfrac{a}{\prod_{k=1}^{j} (1 - |\delta_k|^2)} \right) \right) \\ \phi_{j+1,\Im} = a|\xi| + \displaystyle\sum_{k=1}^{j} \frac{-2|\delta_k|^2 \phi_{k,\Im} + \Im\{\delta_k' \overline{\delta_k}\}}{1 - |\delta_k|^2} \\ b_{j+1} = \dfrac{b_j |\delta_j|^2 - \delta_j'}{1 - |\delta_j|^2} \qquad (j = 0, \cdots, m-1) \end{cases}$$

Symbol class in Z_H

$$f(t,\xi) \in S\{p,q\} \;\Leftrightarrow\; |\partial_t^k f(t,\xi))| \le C_k |\xi|^p (1+t)^{-q-k\beta}$$

Properties:

(i) $f \in S\{p,q\},\ g \in S\{p',q'\} \Rightarrow fg \in S\{p+p',q+q'\}$

(ii) $f \in S\{p,q\} \Rightarrow \partial_t^k f \in S\{p,q+k\}$

(iii) $f \in S\{p,q\} \Rightarrow f \in S\{p+1,q-1\}$

$$a \in S\{0,0\}$$

$$b_1 = -\frac{a'}{2a} \in S\{0,1\}, \quad \delta_1 = \frac{-ib_1}{2\phi_{1,\Im}} = \frac{-ib_1}{2a|\xi|} \in S\{-1,1\}$$

$$b_2 = \frac{b_1|\delta_1|^2 - \delta_1'}{1-|\delta_1|^2} \in S\{-2,3\} \cup S\{-1,2\} = S\{-1,2\}$$

$$b_k \in S\{-k+1,k\},\ \delta_k \in S\{-k,k\} \;\Rightarrow\; b_{k+1} \in S\{-k,k+1\}$$

Proposition.

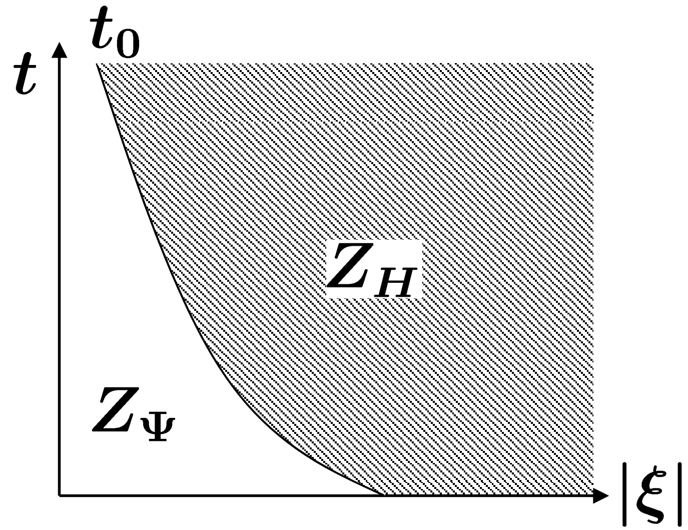
$b_m \in S\{-m+1, m\}$, $|\int_{t_0}^t \phi_m \Re ds| \lesssim 1$, $|V_m(t,\xi)|^2 \simeq \mathcal{E}(t,\xi)$ *in* Z_H.

$$\int_{t_0}^t |b_m|\, ds \lesssim |\xi|^{-m+1} \int_{t_0}^t (1+s)^{-m\beta} ds \lesssim |\xi|^{-m+1}(1+t_0)^{-m\beta+1}$$

$$\leq N^{-m+1}(1+t_0)^{\alpha(m-1)-m\beta+1} \lesssim 1$$

$$\Leftarrow \ \beta \geq \alpha + \frac{1-\alpha}{m}$$

$$C^{-1}\mathcal{E}(t_0,\xi) \leq \mathcal{E}(t,\xi) \leq C\mathcal{E}(t_0,\xi) \quad \text{in } Z_H$$



$$Z_\Psi := \{(t,\xi)\,;\ (1+t)^\alpha|\xi| < N\}$$

$$Z_H := \{(t,\xi)\,;\ (1+t)^\alpha|\xi| \geq N\}$$

$$(1+t_0)^\alpha|\xi| = N$$

<u>$(t,\xi) \in Z_\Psi$</u>

$$\boxed{\tilde{\mathcal{E}}(t,\xi) := \frac{1}{2}\left(a_\infty^2|\xi|^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right)}$$

$$\partial_t\tilde{\mathcal{E}}(t,\xi) = \left(a_\infty^2 - a^2\right)|\xi|^2\Re\{v\overline{v_t}\} \begin{cases} \leq C|a-a_\infty||\xi|\tilde{\mathcal{E}}(t,\xi) \\ \geq -C|a-a_\infty||\xi|\tilde{\mathcal{E}}(t,\xi) \end{cases}$$

$$\Rightarrow\ \tilde{\mathcal{E}}(t,\xi) \begin{cases} \leq \tilde{\mathcal{E}}(0,\xi)\exp\left(C|\xi|\int_0^t|a(s)-a_\infty|ds\right) \leq \tilde{\mathcal{E}}(0,\xi)\exp\left(C\right) \\ \geq \tilde{\mathcal{E}}(0,\xi)\exp\left(-C\right) \end{cases}$$

Summary of the conditions for (GEC)

$$|a'(t)| \le C(1+t)^{-\beta}$$
$$\left(\text{ with } |a^{(k)}(t)| \le C_k(1+t)^{-k\beta} \ (k = 2, \cdots, m)\right)$$
$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \ \ (t \to \infty)$$ ***(Stabilization property)***

Regularity of a

Singular ⟷ ***Regular***

m	1	2	$\cdots$	m	$\cdots$	∞
β	1	$\alpha + \frac{1-\alpha}{2}$	$\cdots$	$\alpha + \frac{1-\alpha}{m}$	$\cdots$	α

Fast ⟷ ***Slow***

Decay order of $|a'(t)|$

$(0 \le \alpha < 1)$

5. Gevrey property

Consider near the critical case $\alpha = \beta$:

$$|a^{(k)}(t)| \le C_k(1+t)^{-k\beta} \ (k = 1, \cdots, m)$$

$$m \to \infty, \beta \to \alpha$$

$$|a^{(k)}(t)| \le Ck!^{\nu} \left((1+t)^{\alpha} (\log(e+t))^{\delta}\right)^{-k} \ (\nu \ge 1, \ \delta \ge 0)$$

Gevrey class $\nu > 1,\ \rho > 0$

$$\gamma_\rho^\nu := \left\{ f(t) \in C^\infty\ ;\ |f^{(k)}(t)| \le C\rho^{-k}\, k!^\nu \right\},\ \gamma^\nu := \bigcup_{\rho>0} \gamma_\rho^\nu$$

Theorem 5.1. ([H.]) $a \in \gamma^\nu\ (\nu > 1)$,

$$|a^{(k)}(t)| \le Ck!^\nu \left((1+t)^\alpha \left(\log(e+t)\right)^\delta \right)^{-k} \quad (\delta \ge 0)$$

$$\exists \alpha \in [0,1), \quad \int_0^t |a(s) - a_\infty| ds = O(t^\alpha) \quad (t \to \infty)$$

$\Rightarrow$ *(GEC):* $E(t) \simeq E(0)$ *is valid hold for* $\delta \ge \nu$*.*

Summary of the conditions for (GEC)

$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha)$$

$$|a^{(k)}(t)| \le C_k \lambda(t)^{-k} \ (k = 1, \cdots, m)$$

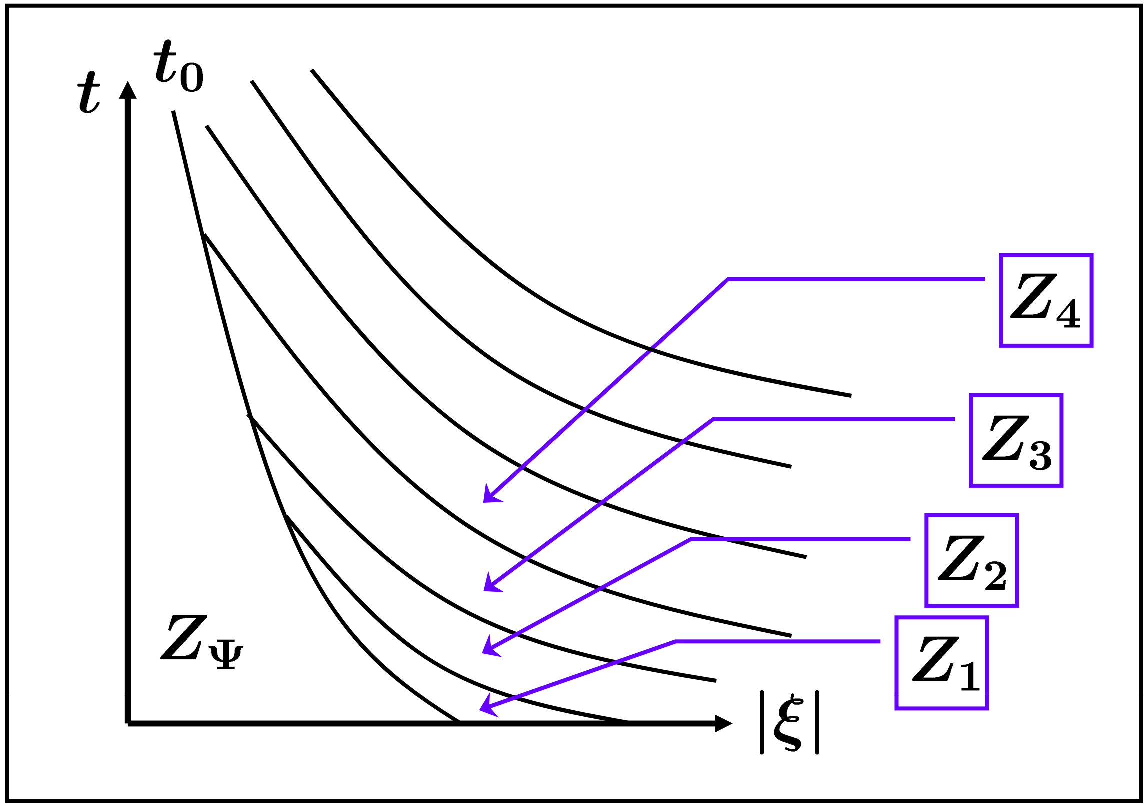
	$m < \infty$		$m = \infty$		
				$C_k = Ck!^\nu$	
$a(t)$	C^1	C^m	C^∞	γ^ν	C^ω
$\lambda(t)$	$t^{1+\varepsilon}$	$t^{\alpha + \frac{1-\alpha}{m}}$	$t^{\alpha+\varepsilon}$	$t^\alpha (\log t)^\nu$	$t^\alpha \log t$

$(0 \le \alpha < 1, \ \nu > 1)$

6. Key idea of the proof of Theorem 5.1

$$t_k(\xi)\ :\ \lambda(t_k)|\xi| = (1+t_k)^{\alpha}\left(\log(e+t_k)\right)^{\delta}|\xi| = N(k+1)^{\nu}$$

$$\begin{aligned} Z_k &:= \{(t,\xi)\ ;\ t_{k-1} < t \le t_k\} \\ &= \{(t,\xi)\ ;\ Nk^{\nu} \le \lambda(t)|\xi| \le N(k+1)^{\nu}\} \end{aligned}$$



Symbol class in Z_k

$$f \in S_k\{p, q; \rho\} \;\Leftrightarrow\; |\partial_t^j f| \le \rho |\xi|^p (q+j)!^\nu \lambda(t)^{-(q+j)}$$

Properties:

(i) $f \in S_k\{p, q; 1\},\ g \in S_k\{p', q'; 1\} \Rightarrow fg \in S_k\{p+p', q+q'; \kappa\}$

(ii) $f \in S_k\{p, q; 1\} \Rightarrow f \in S_k\{p+1, q-1; \frac{1}{N}(\frac{p+k}{k})^\nu\}$

The following estimate is essential for (i):

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{k!(n-k)!}{n!}\right)^\nu \le C \;\;\Leftrightarrow\;\; \nu > 1$$

Proposition. The followings hold uniformly with respect to k:
$|\int_{t_0}^{t} \phi_{k\Re} ds| \lesssim 1,\ |V_k(t,\xi)|^2 \simeq \mathcal{E}(t,\xi),\ b_k \in S_k\{-k+1, k; \rho^k\}\ \mathit{in}\ Z_k.$

$$b_k \in S_k\{-k+1, k; \rho^k\} \ \Rightarrow\ |b_k| \le \rho^k k!^\nu \lambda(t)^{-k} |\xi|^{-k+1}$$

$$\begin{aligned}(t,\xi) \in Z_k\ &\Leftrightarrow N k^\nu \le \lambda(t)|\xi| \le N(k+1)^\nu \\ &\Leftrightarrow \rho^k k!^\nu \lambda(t)^{-k}|\xi|^{-k+1} \\ &\qquad \le \begin{cases}\rho^{k-1}(k-1)!^\nu \lambda(t)^{-k+1}|\xi|^{-k+2} \\ \rho^{k+1}(k+1)!^\nu \lambda(t)^{-k-1}|\xi|^{-k} \qquad (\rho = N)\end{cases}\end{aligned}$$

The order of b_k is better than b_{k-1} and b_{k+1} in Z_k.

$$\int_{t_{k-1}}^{t} |b_k|\, ds \le \rho^k k!^\nu (|\xi|\lambda(t_{k-1}))^{-k} t_k |\xi| = \rho^k k!^\nu (N k^\nu)^{-k} t_k |\xi|$$

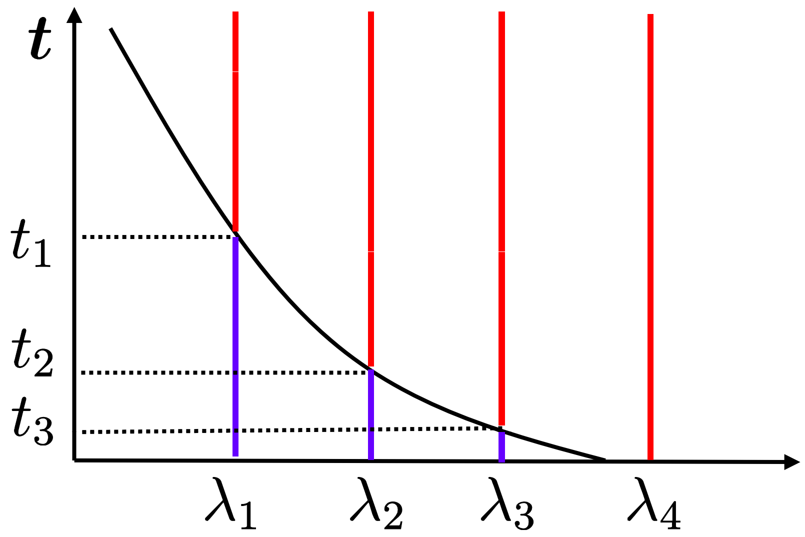
$$= \left(\tfrac{\rho}{N}\right)^k \left(\tfrac{k!}{k^k}\right)^\nu t_k |\xi| \lesssim (\log t)^\nu$$

6. Some remarks

(I) Initial boundary value problem with a compact boundary

$$\begin{cases} (\partial_t^2 - a(t)^2\Delta)\, u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \Omega, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \Omega, \\ u(t,x) = 0, \quad (t,x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

The first eigenvalue of $-\Delta$ is positive $\Rightarrow Z_\Psi$ is compact



$$(1+t_j)^\alpha \lambda_j = N \quad (j = 1, 2, \cdots)$$

$$Z_{\Psi,j} := [0, \max\{t_j, 0\})$$

$$Z_{H,j} := [\max\{0, t_j\}, \infty)$$

(GEC) for IBVP in bounded daomin

$$\int_0^t |a(s) - a_\infty| ds = O(t^\alpha), \quad |a^{(k)}(t)| \le C_k \lambda(t)^{-k} \ (k = 1, \cdots, m)$$

$$(0 \le \alpha < 1, \ \nu > 1)$$

Cauchy Problem

	$m < \infty$		$m = \infty$		
				$C_k = Ck!^\nu$	
$a(t)$	C^1	C^m	C^∞	γ^ν	C^ω
$\lambda(t)$	$t^{1+\varepsilon}$	$t^{\alpha + \frac{1-\alpha}{m}}$	$t^{\alpha+\varepsilon}$	$t^\alpha (\log t)^\nu$	$t^\alpha \log t$

IBVP

	$m < \infty$		$m = \infty$		
				$C_k = Ck!^\nu$	
$a(t)$	C^1	C^m	C^∞	γ^ν	C^ω
$\lambda(t)$	$t^{1+\varepsilon}$	$t^{\frac{1}{m}}$	t^ε	$(\log t)^\nu$	$\log t$

(II) Application to the Global solvability of Kirchhoff equation

$$\partial_t^2 u - \left(1 + \|\nabla u(t,\cdot)\|_{L^2}^2\right) \Delta u = 0 \quad \text{(Kirchhoff equation)}$$

$$\Downarrow$$

$$\partial_t^2 u - a(t)^2 \Delta u = 0 \quad \text{(Linearized equation)}$$

(i) We easily have the estimate:

$$|a'(t)| \leq C(T-t)^{-\frac{3}{2}} \quad (\exists T > 0) \;\Rightarrow\; \exists \text{local solution on } [0,T)$$

(ii) The solution can be prolonged from $t = T$ if $|a'(t)| \in L^1([0,T))$,

but one cannot expect in in general…

(iii) $|a'(t)| \in L^1([0,T))$ is not necessary for $\sup_{t\in[0,T)}\{E_1(t)\} < \infty$

if $a^{(k)}(t)$ satisfy some suitable conditions on [0,T).

[Manfrin (2005)]

Global solvability in B^3_Δ by using C^3 properties of $a(t)$.

[H. (2006)]

Global solvability in B^m_Δ $(\forall m \geq 4)$ by using C^m properties of $a(t)$.

(III) Weakly hyperbolic equation with Gevrey coefficients

[Colombini – Nishitani (2000)]

[H. (2005)] (C^∞ coefficient)