On the energy estimates of second order hyperbolic equations with time dependent coefficients

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§ 1. Introduction

The Cauchy problem of 2nd order hyperbolic equation:

$$
(\mathrm{C}) \begin{cases} (\partial_t^2 + 2b\partial_t\partial_x - a^2\partial_x^2)\, u = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)) \ \ x \in \mathbb{R}, \\ a = a(t),\ b = b(t),\ a^2 + b^2 > 0. \end{cases}
$$

Total energy of the solution of (C):

$$
E(t) := \|\partial u(t,\cdot)\|_{L^2(\mathbb{R}_x)}^2 + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R}_x)}^2
$$

"Generalized Energy Conservation (=(GEC)" as $t \to \infty$:

$$
C^{-1}E(0) \le E(t) \le CE(0) \ \Leftrightarrow \ E(t) \simeq E(0).
$$

Basic properties

$$E_0(t) := a(t)^2\|\partial u(t,\cdot)\|^2 + \|\partial_t u(t,\cdot)\|^2 \simeq E(t),$$

$$a \in C^1([0,\infty)),\ 0 < a_0 \le a(t) \le a_1.$$

$$\begin{aligned} E_0'(t) =& 2a'(t)a(t)\|\partial u(t,\cdot)\|^2 - 4b(t)\left(\partial\partial_t u(t,\cdot), \partial_t u(t,\cdot)\right) \\ &\lesseqgtr \pm 2a_0^{-1}|a'(t)|E_0(t) \\ &\Rightarrow E_0(t) \lesseqgtr \exp\left(\pm 2a_0^{-1}\int_0^t |a'(s)|ds\right) E_0(0) \end{aligned}$$

(i) $b(t)$ has no influence for the estimate of $E_0(t)$;

(ii) No "energy conservation" if $a'(t) \not\equiv 0$;

(iii) $a' \in L^1(\mathbb{R}_+) \ \Rightarrow\ E_0(t) \simeq E_0(0)$ (= (GEC));

(iv) (GEC) is not trivial if $a' \notin L^1(\mathbb{R}_+)$.

§ 2. Previous results

Conditions to the coefficients

$$a(t),\ b(t) \in C^m([0,\infty)),\ m \geq 1,$$

$$0 < c_0 \leq \sqrt{a(t)^2 + b(t)^2} =: c(t) \leq c_1,$$

$$\begin{cases} |a^{(k)}(t)| \lesssim (1+t)^{-k\beta_a} \\ |b^{(k)}(t)| \lesssim (1+t)^{-k\beta_b} \end{cases} \qquad (k = 1, \cdots, m)$$

$$\int_0^t (|a(s) - a_\infty| + |b(s) - b_\infty|)\, ds \lesssim (1+t)^\alpha$$

$$(\exists a_\infty,\ \exists b_\infty,\ \alpha \in [0,1))$$

(i) (GEC) is trivial if $\beta_a > 1$;

(ii) $\alpha = 1$ gives a trivial condition.

$\underline{b(t) \equiv 0}$

$$a(t) \in C^m,\ |a^{(k)}| \lesssim t^{-k\beta_a},\ \int_0^t |a - a_\infty| ds \lesssim t^\alpha$$

$m = 1,\ \beta_a > 1 \Rightarrow$ (GEC) (trivial case)

$m = 2,\ \beta_a = 1 \Rightarrow$ (GEC) (Yamazaki ('89), Reissig-Smith ('05))

$m = \infty,\ \beta_a < 1 \Rightarrow$ *not* (GEC) (Reissig-Smith ('05))

$m \geq 2,\ \alpha < 1,\ \beta_a \geq \alpha + \frac{1-\alpha}{m} (< 1) \Rightarrow$ (GEC) (H. ('07))

$m = \infty,\ \beta_a < \alpha < 1 \Rightarrow$ *non* (GEC) (H. ('07))

$a \in \gamma^{(s)}$ (*Gevrey class*), $\beta_a \to \alpha + 0$ (H. ('10))

$b(t) \not\equiv 0$

$$a(t), b(t) \in C^m, \; |a^{(k)}| \lesssim t^{-k\beta_a}, \; |b^{(k)}| \lesssim t^{-k\beta_b},$$
$$\int_0^t (|a - a_\infty| + |b - b_\infty|)\, ds \lesssim t^\alpha$$

$m = 1, \; \beta_a > 1 \Rightarrow$ (GEC) (trivial case)

$m = 2, \; \beta_a = 1, \; \beta_b > 1 \Rightarrow$ (GEC) (cf. Reissig-H. ('05))

$m = 2, \; \beta_a = \beta_b = 1 \Rightarrow$ *non* (GEC) (cf. Reissig-H. ('04))

$m = 2, \; \beta_a = \beta_b = 1,$ "Levi's condition" (L1):

(L1) $\left| \int_0^t \frac{b'(s)}{c(s)}\, ds \right| \leq C \;\Rightarrow$ (GEC) (cf. Reissig-H. ('05))

Summary of the previous results

$$(\partial_t^2 + 2b\partial_t\partial_x - a^2\partial_x^2)u = 0, \quad c = \sqrt{a^2 + b^2},$$

$$|a^{(k)}| \lesssim t^{-k\beta_a}, \ |b^{(k)}| \lesssim t^{-k\beta_b}, \qquad \left| \int_0^t \frac{b'(s)}{c(s)}\, ds \right| \le C$$

$$\int_0^t (|a - a_\infty| + |b - b_\infty|)ds \lesssim t^\alpha. \qquad \text{(L1)}$$

	$b(t) \equiv 0$	$b(t) \not\equiv 0$
$m = 1$	$\beta_a > 1$	
$m = 2$	$\beta_a = 1$	$\beta_a = \beta_b = 1$, (L1) ($\beta_a = \beta_b = 1$, non (L1))
$m \ge 2$	$\beta_a \ge \alpha + \frac{1-\alpha}{m}$ ($\beta_a < \alpha$)	? ? ?

Examples

$\int_0^t |a(s) - a_\infty| \, ds \lesssim t^\alpha$

$$\chi(\tau) \in C^m([0,1]),\ \chi(0) = \chi(1) = 1,\ \chi(\tau) \geq 0,$$

$$I_j = [t_j, t_j + t_j^\alpha],\ t_j = 2^j,$$

$$a(t) = \begin{cases} \chi(t_j^{-\alpha}(t - t_j)) & (t \in I_j,\ j = 1, 2, \cdots) \\ 1 & (t \notin \bigcup_j I_j) \end{cases}$$

$\left| \int_0^t \frac{b'(s)}{c(s)} \, ds \right| \leq C$

$$a(t) = b(t) \ \Rightarrow \ \left| \int_0^t \frac{b'(s)}{c(s)} \, ds \right| = \frac{1}{\sqrt{2}} \left| \int_0^t \frac{a'(s)}{a(s)} \, ds \right|$$

Why do we call (L1) "Levi's condition"?

$$\left(\partial_t^2 + 2b\partial_t\partial_x - a^2\partial_x^2\right) u = 0$$

$$\Downarrow \quad v(t;\xi) = \hat{u}(t,\xi)$$

$$v'' + 2\mathrm{i}b\xi v' + a^2\xi^2 v = 0$$

$$\Downarrow \quad w(t;\xi) = \exp\left(-\mathrm{i}\xi \int_\tau^t b(s)\,ds\right) v(t;\xi)$$

$$w'' + c^2\xi^2 w - \mathrm{i}\xi b' w = 0,\ c = \sqrt{a^2+b^2}$$

$c(t) = t^l,\ b'(t) = t^k$

$$\left|\int_t^\infty \frac{b'(s)}{c(s)}\,ds\right| \le C \Leftarrow k > l-1$$

§ 3. Main theorem

$$\left(\partial_t^2 + 2b(t)\partial_t\partial_x - a(t)^2\partial_x^2\right) u = 0$$

$$|a^{(k)}(t)| + |b^{(k)}(t)| \lesssim t^{-k\beta} \ (k = 1, \cdots, m)$$

$$\int_0^t (|a(s) - a_\infty| + |b(s) - b_\infty|)\, ds \lesssim t^\alpha$$

$$c = \sqrt{a^2 + b^2}, \qquad \left|\int_0^t \frac{b'(s)}{c(s)}\, ds\right| \le C \cdots \text{(L1)}$$

<u>Theorem 1</u>

$$m = 2,\ \beta \ge \tfrac{\alpha+1}{2}\left(= \alpha + \tfrac{1-\alpha}{2}\right),\ \text{(L1)} \Rightarrow \text{(GEC)}.$$

$$m = 3,\ \beta \ge \tfrac{2\alpha+1}{3}\left(= \alpha + \tfrac{1-\alpha}{3}\right),\ \text{(L1)} \Rightarrow \text{(GEC)}.$$

Conditions for (GEC)

	$b(t) \equiv 0$	$b(t) \not\equiv 0$
$m = 1$	$\beta_a > 1$	
$m = 2$	$\beta_a = 1$	$\beta_a = \beta_b = 1$, (L1) $(\beta_a = \beta_b = 1$, non (L1)$)$
	$\beta_a \geq \alpha + \frac{1-\alpha}{2}$	$\beta \geq \alpha + \frac{1-\alpha}{2}$, (L1)
$m = 3$	$\beta_a \geq \alpha + \frac{1-\alpha}{3}$	$\beta \geq \alpha + \frac{1-\alpha}{3}$, (L1)
$m \geq 4$	$\beta_a \geq \alpha + \frac{1-\alpha}{m}$ $(\beta_a < \alpha)$	? ? ?

$$\left(\partial_t^2 + 2b(t)\partial_t\partial_x - a(t)^2\partial_x^2\right) u = 0$$

$$|a^{(k)}(t)| + |b^{(k)}(t)| \lesssim t^{-k\beta} \ (k = 1, \cdots, m)$$

$$\int_0^t (|a(s) - a_\infty| + |b(s) - b_\infty|)\, ds \lesssim t^\alpha$$

$$c = \sqrt{a^2 + b^2}, \qquad \left|\int_0^t \frac{b'(s)}{c(s)}\, ds\right| \le C \cdots \text{(L1)}$$

$$\left|\int_t^\infty \frac{c(b'c'' - b''c') + b'((c')^2 - (b')^2)}{c^5} ds\right| \le t^{-2\alpha} \quad \text{(L2)}$$

Theorem 2

$$m = 4, 5, \quad \beta \ge \alpha + \tfrac{1-\alpha}{m}, \text{(L1)}, \text{(L2)} \Rightarrow \text{(GEC)}.$$

Conditions for (GEC)

	$b(t) \equiv 0$	$b(t) \not\equiv 0$
$m = 1$	$\beta_a > 1$	
$m = 2$	$\beta_a = 1$	$\beta_a = \beta_b = 1$, (L1) ($\beta_a = \beta_b = 1$, non (L1))
$m = 2, 3$	$\beta_a \geq \alpha + \frac{1-\alpha}{m}$ ($\beta_a < \alpha$)	$\beta \geq \alpha + \frac{1-\alpha}{m}$, (L1)
$m = 4, 5$		$\beta \geq \alpha + \frac{1-\alpha}{m}$, (L1), (L2)
$m \geq 6$		? ? ?

Remarks

$$\boxed{|a^{(k)}(t)| + |b^{(k)}(t)| \lesssim t^{-k\beta} \ (k = 1, \cdots, m)}$$

I. (L1) is not trivial for $\beta = \alpha + \frac{1-\alpha}{m} (< 1)$, $m \geq 2$.

$$\left| \int_0^t \frac{b'(s)}{c(s)}\, ds \right| \leq \int_0^t \frac{|b'(s)|}{c_0}\, ds \lesssim \int_0^t (1+s)^{-\beta}\, ds \to \infty$$

II. (L2) is not trivial for $\beta = \alpha + \frac{1-\alpha}{m} (< \frac{2\alpha+1}{3})$, $m \geq 4$.

$$\left| \int_t^\infty \frac{c(b'c'' - b''c') + b'((c')^2 - (b')^2)}{c^5} ds \right|$$

$$\lesssim \int_0^t (1+s)^{-3\beta}\, ds \lesssim t^{1-3\beta} \ (\lesssim t^{-2\alpha} \text{ is not true})$$

§ 4. Sketch of the proof

Our goal: prove the estimate

$$\mathcal{E}(t;\xi) = |v'(t;\xi)|^2 + \xi^2 |v(t;\xi)|^2 \simeq \mathcal{E}(0;\xi)$$

$$v'' + 2\mathrm{i}b\xi v' + a^2\xi^2 v = 0$$

<u>In low frequency part</u>

We easliy have the estimate $\mathcal{E}(t;\xi) \simeq \mathcal{E}(0;\xi)$ in Z_Ψ:

$$Z_\Psi = \{(t,\xi)\ ;\ 0 \le t < t_\xi\},\ (1+t_\xi)^\alpha |\xi| = N$$

by

$$\int_0^t (|a(s) - a_\infty| + |b(s) - b_\infty|)\ ds \lesssim t^\alpha.$$

In high frequency part

$$Z_H = \{(t,\xi)\ ;\ t \geq t_\xi\},\ (1+t_\xi)^\alpha|\xi| = N$$

Keywords:
symbol class, diagonalization, real parts of diagonal entries

$$f(t,\xi) \in S\{p,q\} \ \Leftrightarrow\ |\partial_t^k f(t,\xi)| \lesssim |\xi|^p (1+t)^{-\beta(k+q)}$$

$$S\{p,q\} \supset S\{p-1,q+1\}$$

$$f(t,\xi) \in S\{-(m-1),m\}$$

$$\Rightarrow\ \int_{t_\xi}^t |f(s,\xi)|\,ds \lesssim C \text{ for } \beta \geq \alpha + \tfrac{1-\alpha}{m}$$

$$w'' + a^2\xi^2 w - \mathrm{i}b'\xi w = 0, \quad w(t;\xi) = e^{-\mathrm{i}\xi\int_\tau^t b(s)\,ds} v(t;\xi)$$

$$\Downarrow$$

$$\left(\tfrac{d}{dt} - \Phi_1 - R_1\right) W_1 = 0,\ W_1 = {}^t(w' + \mathrm{i}\xi c w, w' - \mathrm{i}\xi c w)$$

$$\Phi_1 = \begin{pmatrix} \phi_{1+} & 0 \\ 0 & \phi_{1-} \end{pmatrix}, \quad \phi_{1\pm} = \pm\mathrm{i}\xi c - \frac{c'}{2c} \mp \frac{b'}{2c}$$

$$R_1 = \begin{pmatrix} 0 & r_{1+} \\ r_{1-} & 0 \end{pmatrix}, \quad r_{1\pm} = -\frac{c'}{2c} \mp \frac{b'}{2c} \in S\{0,1\}$$

$$\begin{aligned} |W_1(t;\xi)|^2 &\simeq \mathcal{E}(t;\xi) \\ &\lesssim \mathcal{E}(t_\xi;\xi)\exp\left(C\left(\left|\int_{t_\xi}^t \Re\{\phi_{1\pm}\}ds\right| + \int_{t_\xi}^t |r_{1\pm}|ds\right)\right) \end{aligned}$$

Without (L1)

$$\left(\frac{d}{dt} - \Phi_1 - R_1\right) W_1 = 0,\ R_1 \in S\{0,1\}$$

$a, b \in C^2$ ⇩ $\exists M_1$ in Z_H, $W_2 = M_1 W_1$,

$$\left(\frac{d}{dt} - \Phi_2 - R_2\right) W_2 = 0,$$

$$R_2 \in S\{0,1\}\ (R_2 \in S\{-1,2\} \text{ for } b \equiv 0)$$

$a, b \in C^3$ ⇩ $\exists M_2$ in Z_H, $W_3 = M_2 W_2$,

$$\left(\frac{d}{dt} - \Phi_3 - R_3\right) W_3 = 0,$$

$$R_3 \in S\{0,1\}\ (R_3 \in S\{-2,3\} \text{ for } b \equiv 0)$$

With (L1)

$$\left(\frac{d}{dt} - \Phi_1 - R_1\right) W_1 = 0,\ R_1 \in S\{0,1\}$$

$a, b \in C^2 \Downarrow \exists M_1$ in Z_H, $W_2 = M_1 W_1$,

$$\left(\frac{d}{dt} - \Phi_2 - R_2\right) W_2 = 0,$$
$$R_2 \in S\{-1,2\},\ \left|\exp\left(\int_{t_\xi}^{t} \phi_{2\pm}(s,\xi)\right)\, ds\right| \le C$$

$a, b \in C^3 \Downarrow \exists M_2$ in Z_H, $W_3 = M_2 W_2$,

$$\left(\frac{d}{dt} - \Phi_3 - R_3\right) W_3 = 0,$$
$$R_3 \in S\{-2,3\},\ \left|\exp\left(\int_{t_\xi}^{t} \phi_{3\pm}(s,\xi)\right)\, ds\right| \le C$$

$$|W_1(t;\xi)|^2 \simeq |W_2(t;\xi)|^2 \simeq |W_3(t;\xi)|^2 \simeq \mathcal{E}(t;\xi)$$

With (L1) and (L2)

$$\left(\frac{d}{dt} - \Phi_3 - R_3\right) W_3 = 0,\ R_3 \in S\{-2,3\}$$

$a, b \in C^4 \Downarrow$ $\exists M_3$ in Z_H, $W_4 = M_3 W_3$,

$$\left(\frac{d}{dt} - \Phi_4 - R_4\right) W_4 = 0,$$

$$R_4 \in S\{-3,4\},\ \left|\exp\left(\int_{t_\xi}^{t} \phi_{4\pm}(s,\xi)\right) ds\right| \le C$$

$a, b \in C^5 \Downarrow$ $\exists M_4$ in Z_H, $W_5 = M_4 W_4$,

$$\left(\frac{d}{dt} - \Phi_5 - R_5\right) W_5 = 0,$$

$$R_5 \in S\{-4,5\},\ \left|\exp\left(\int_{t_\xi}^{t} \phi_{5\pm}(s,\xi)\right) ds\right| \le C$$

$$|W_4(t;\xi)|^2 \simeq |W_5(t;\xi)|^2 \simeq \mathcal{E}(t;\xi)$$

Thank you very much!