Energy estimates and diagonalization for hyperbolic equations with time dependent coefficients

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This note is based on the lecture in the research seminar "Complex Analysis and Partial Differential Equation" from January 9 to 13, 2012 at Hanoi University of Science and Technology. The aim of this lecture is to introduce the collections of some results and techniques for the energy estimates of hyperbolic equations with time dependent coefficients. The main tools for the proof of the theorems in this note are based on Fourier analysis, but the details are omitted. Therefore, the reader should occasionally refer the textbooks for Fourier analysis and partial differential equation, for instance [18]. The author would like to grateful to Professor Le Hung Son and the participants in the lecture for their kindly hospitality during the stay at Hanoi.

Contents

1 Fourier transformation

1.1 Definition of the Fourier transformation

Proposition 1.1 (Fourier series)**.** *Let $L > 0$ and $f(x)$ be a $2L$-periodic function. If $f(x) \in L^2((-L, L))$, then $f(x)$ is represented by the following trigonometric series:*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{\mathrm{i}\pi n x}{L}\right), \tag{1.1}$$

where

$$c_n = \frac{1}{2L}\int_{-L}^{L} \exp\left(-\frac{\mathrm{i}\pi n x}{L}\right) f(x)\, dx. \tag{1.2}$$

Let us denote

$$k_n = \frac{\pi n}{L}, \quad F(k_n) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L} \exp\left(-\mathrm{i}k_n x\right) f(x)\, dx, \tag{1.3}$$

it follows that

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{\pi}{L} F(k_n) \exp\left(\mathrm{i}x k_n\right). \tag{1.4}$$

Here $F(k_n)$ describes the spectral strength. Then we consider the limit $L \to \infty$ as follows:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \frac{\pi}{L} F(k_n) \exp\left(\mathrm{i}x k_n\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) \exp\left(\mathrm{i}x\xi\right) d\xi \quad (L \to \infty).$$

Actually, $f(x) = (f(x-0) + f(x+0))/2$ if $f(x)$ is not continuous at $x = 0$.

Definition 1.1 (Fourier transformation and inverse transformation)**.** Let $f(x) \in L^2(\mathbb{R})$. We define the Fourier transformation of $f(x)$ by

$$\mathscr{F}[f(x)](\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} f(x)\, dx = F(\xi). \tag{1.5}$$

On the other hand, for $F(\xi) \in L^2(\mathbb{R})$ we define the inverse transformation of $F(\xi)$ by

$$\mathscr{F}^{-1}[F(\xi)](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi} F(\xi)\, dx = f(x). \tag{1.6}$$

Proposition 1.2. *$\mathscr{F}^{-1}\left[\mathscr{F}[f(x)](\xi)\right](x) = f(x)$.*

1.2 Some properties

Let us denote $\mathscr{F}[f(x)](\xi) = F(\xi)$ and $\mathscr{F}[g(x)] = G(\xi)$. The following properties are established:

(i) $\mathscr{F}[f(-x)](\xi) = F(-\xi)$;

(ii) $\mathscr{F}[af(x) + bg(x)](\xi) = aF(\xi) + bG(\xi)$;

(iii) $\mathscr{F}[f(x-a)](\xi) = \mathrm{e}^{-\mathrm{i}a\xi} F(\xi)$;

(iv) $\mathscr{F}[f(ax)](\xi) = \frac{1}{|a|} F(\frac{\xi}{a})$;

(v) $\mathscr{F}[x^n f(x)](\xi) = \mathrm{i}^n F^{(n)}(\xi)$;

(vi) $\mathscr{F}[f^{(n)}(x)](\xi) = (\mathrm{i}\xi)^n F(\xi)$;

(vii) $\mathscr{F}[\int_{-\infty}^{x} f(y) dy](\xi) = \frac{1}{\mathrm{i}\xi} F(\xi)$;

(viii) $\mathscr{F}[(f*g)(x)](\xi) = F(\xi)G(\xi)$, $(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(y)g(x-y)dy$.

Example 1.1.

$$\mathscr{F}\left[\exp\left(-ax^2\right)\right](\xi) = \frac{1}{\sqrt{2a}}\exp\left(-\frac{\xi^2}{4a}\right).$$

Proof.

$$\begin{aligned}\mathscr{F}\left[\exp\left(-ax^2\right)\right](\xi) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi - ax^2}dx = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\xi^2}{4a}}\int_{-\infty}^{\infty}\exp\left(-a\left(x+\frac{\mathrm{i}\xi}{2a}\right)^2\right)dx \\ =& \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\xi^2}{4a}\right)\int_{-\infty}^{\infty}\exp\left(-ax^2\right)dx = \frac{1}{\sqrt{2\pi a}}\exp\left(-\frac{\xi^2}{4a}\right)\int_{-\infty}^{\infty}\exp\left(-y^2\right)dx \\ =& \frac{1}{\sqrt{2a}}\exp\left(-\frac{\xi^2}{4a}\right).\end{aligned}$$

□

$$\mathscr{F}\left[\frac{a}{a^2+x^2}\right](\xi) = \pi\exp\left(-2\pi a\xi\right).$$

$$\mathscr{F}\left[\exp\left(-\frac{|x|}{a}\right)\right](\xi) = \frac{2r}{1+4\pi^2a^2\xi^2}.$$

$$f(x) = \begin{cases}\frac{1}{a} & |x| < \frac{a}{2}, \\ 0 & |x| \geq \frac{a}{2},\end{cases} \quad \mathscr{F}[f(x)](\xi) = \frac{\sin(\pi a\xi)}{\pi a\xi}.$$

Proposition 1.3 (Parseval's identity)**.** *Let $f, g \in L^2(\mathbb{R})$. Then the following identity is established:*

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\infty}^{\infty}|F(\xi)|^2dx. \tag{1.7}$$

Proof. Let us define $g(x) = \int_{-\infty}^{\infty} f(x+y)\overline{f(y)}dy$. Then we have

$$\begin{aligned}G(\xi) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}x\xi}\int_{-\infty}^{\infty}f(x+y)\overline{f(y)}dy\,dx \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}y\xi}\overline{f(y)}\left(\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}(x+y)\xi}f(x+y)dx\right)dy \\ =& F(\xi)\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}y\xi}\overline{f(y)}dy = \sqrt{2\pi}F(\xi)\overline{F(\xi)}.\end{aligned}$$

It follows that

$$\int_{-\infty}^{\infty}f(x+y)\overline{f(y)}dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}x\xi}G(\xi)\,d\xi = \int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}x\xi}F(\xi)\overline{F(\xi)}\,d\xi.$$

Therefore, as $x \to 0$ we have

$$\int_{-\infty}^{\infty}f(x)\overline{f(y)}dy = \int_{-\infty}^{\infty}F(\xi)\overline{F(\xi)}dy. \tag{1.8}$$

□

1.3 Applications of Fourier transformation to PDE

Wave equation. Let us consider the following in initial value problem for the wave equation:

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.9}$$

where $a > 0$ describes the propagation speed of the wave. By partial Fourier transformation with respect to x we have

$$\begin{cases} v_{tt} + a^2 \xi^2 v = 0, \quad (t,\xi) \in (0,\infty) \times \mathbb{R}, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) = v_0(\xi), \quad \hat{u}_t(0,\xi) = \hat{u}_1(\xi) = v_1(\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{1.10}$$

where $v(t,\xi) = \hat{u}(t,\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} u(t,x)\,dx$. Then the solution to (1.10) is given as follows:

$$v(t,\xi) = \frac{1}{2}\left(v_0(\xi) + \frac{v_1(\xi)}{\mathrm{i}a\xi}\right)\mathrm{e}^{\mathrm{i}a\xi t} + \frac{1}{2}\left(v_0(\xi) - \frac{v_1(\xi)}{\mathrm{i}a\xi}\right)\mathrm{e}^{-\mathrm{i}a\xi t}. \tag{1.11}$$

Therefore, by inverse Fourier transformation, the solution to the original problem (1.9) is given by

$$u(t,x) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi}\left(\frac{1}{2}\left(v_0(\xi) + \frac{v_1(\xi)}{\mathrm{i}a\xi}\right)\mathrm{e}^{\mathrm{i}a\xi t} + \frac{1}{2}\left(v_0(\xi) - \frac{v_1(\xi)}{\mathrm{i}a\xi}\right)\mathrm{e}^{-\mathrm{i}a\xi t}\right) d\xi. \tag{1.12}$$

Recalling the formulas $\mathscr{F}^{-1}[F(\xi)\mathrm{e}^{\mathrm{i}a\xi t}](x) = f(x+at)$ and

$$\mathscr{F}^{-1}\left[\frac{v_1(\xi)}{\mathrm{i}a\xi}\right](\xi) = \frac{1}{a}\int_{-\infty}^{x} \mathscr{F}^{-1}[v_1(\xi)](y)dy = \frac{1}{a}\int_{-\infty}^{x} u_1(y)dy,$$

we have

$$u(t,x) = \frac{1}{2}\left(u_0(x+at) + u_0(x-at) + \frac{1}{a}\int_{x-at}^{x+at} u_1(y)dy\right). \tag{1.13}$$

Heat equation. Let us consider the following in initial value problem for the heat equation:

$$\begin{cases} u_t - c u_{xx} = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.14}$$

where $c > 0$. By partial Fourier transformation with respect to x we have

$$\begin{cases} v_t + c\xi^2 v = 0, \quad (t,\xi) \in (0,\infty) \times \mathbb{R}, \\ \hat{u}(0,\xi) = \hat{u}_0(\xi) = v_0(\xi), \quad \xi \in \mathbb{R}. \end{cases} \tag{1.15}$$

Therefore, the solution to (1.15) is given by

$$v(t,\xi) = v_0(\xi)\exp\left(-c\xi^2 t\right). \tag{1.16}$$

Thus the solution to the original problem is represented as follows:

$$\begin{aligned} u(t,x) =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\xi} v_0(\xi)\exp\left(-c\xi^2 t\right) d\xi \\ =& \mathscr{F}^{-1}\left[v_0(\xi)\exp\left(-c\xi^2 t\right)\right](x) \\ =& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_0(y)\mathscr{F}^{-1}\left[\exp\left(-c\xi^2 t\right)\right](y-x)dy \\ =& \frac{1}{2\sqrt{\pi c t}}\int_{-\infty}^{\infty} u_0(y)\exp\left(-\frac{(x-y)^2}{4ct}\right) dy. \end{aligned}$$

Schrödinger equation. Let us consider the following in initial value problem for the Schrödinger equation:

$$\begin{cases} u_t - \mathrm{i}u_{xx} = 0, & (t,x) \in (-\infty,0)\cup(0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.17}$$

Putting $c = \mathrm{i}$ to the solution of the heat equation, the solution of (1.17) is represented as follows:

$$u(t,x) = \frac{1}{2\sqrt{\mathrm{i}\pi t}} \int_{-\infty}^{\infty} u_0(y) \exp\left(-\frac{(x-y)^2}{4\mathrm{i}t}\right) dy. \tag{1.18}$$

1.4 Energy conservation

Let us consider the energy of the solution to the wave equation of (1.9):

$$u_{tt} - a^2 u_{xx} = 0. \tag{1.19}$$

Here the propagation speed a is given by $a = T/\rho$ if u describes the shape of a string whose tension and the density are T, and ρ respectively. Then the total energy of (1.9) at t in physical meaning is given by

$$E_W(t) := \frac{1}{2}\int_{-\infty}^{\infty} |u_t(t,x)|^2 dx + \frac{1}{2}a^2 \int_{-\infty}^{\infty} |u_x(t,x)|^2 dx, \tag{1.20}$$

where the first, and the second terms describe the kinetic energy, and the elastic energy respectively. Then recalling the Parceval Formula we have

$$E_W(t) := \frac{1}{2}\int_{-\infty}^{\infty} \left(|v_t(t,\xi)|^2 + a^2\xi^2|v(t,\xi)|^2\right) d\xi, \tag{1.21}$$

where $v(t,\xi) = \hat{u}(t,\xi)$. Then we have the following property, which is called the energy conservation:

Proposition 1.4. *Let $u(t,x)$ be a solution of (1.9), and $u(t,x) \in C^2([0,\infty); L^2(\mathbb{R}_x))$. Then the energy conservation law: $E_W(t) \equiv E_W(0)$ is established.*

Proof. Differentiating $E_W(t)$ with respect to t we have

$$\begin{aligned} \frac{d}{dt}E_W(t) &= \int_{-\infty}^{\infty} \left(\Re\{u_{tt}\overline{u_t}\} + a^2\Re\{u_{xt}\overline{u_x}\}\right) dx \\ &= \int_{-\infty}^{\infty} a^2\Re\{u_{xx}\overline{u_t}\}dx - \int_{-\infty}^{\infty} a^2\Re\{u_t\overline{u_{xx}}\}dx = 0, \end{aligned}$$

which conclude the energy conservation law. □

Let us assume that the solution to the Cauchy problem (1.9) satisfies $u(t,x) \in C^2([0,\infty); L^2(\mathbb{R}_x))$. Then we can define the following energy of microlocal version to the solution of (1.10) as follows:

$$\mathcal{E}_W(t,\xi) = \frac{1}{2}\left(|v_t(t,\xi)|^2 + a^2\xi^2|v(t,\xi)|^2\right). \tag{1.22}$$

Then we have the following proposition, which give that the energy conservation in each frequency:

Proposition 1.5. *Let $v(t,\xi)$ be a solution of (1.10), and $v(t,\xi) \in C^2([0,\infty); L^\infty(\mathbb{R}_\xi))$. Then the energy conservation law of microlocal version: $\mathcal{E}_W(t,\xi) \equiv \mathcal{E}_W(0,\xi)$ is established.*

Let $v(t,\xi)$ be the solution of (1.15), which is reduced from the Cauchy problem of the heat equation (1.14). If we define $\mathcal{E}_H(t,\xi)$ by

$$\mathcal{E}_H(t,\xi) = \frac{1}{2}|v(t,\xi)|^2, \tag{1.23}$$

then we have

$$\partial_t\mathcal{E}_H(t,\xi) = \Re\{v_t\overline{v}\} = -c\xi^2|v|^2 = -2c\xi\mathcal{E}_H(t,\xi). \tag{1.24}$$

It follows that

$$\mathcal{E}_H(t,\xi)=\mathcal{E}_H(0,\xi)\mathrm{e}^{-2c\xi^2 t}. \tag{1.25}$$

Therefore, we see that $\mathcal{E}_H(t,\xi)$ decays exponential order for $|\xi|>0$. Here we note that the estimate (1.25) does not bring the exponential order decay of the $\|v(t,\cdot)\|_{L^2(\mathbb{R}_x)}=\|u(t,\cdot)\|_{L^2(\mathbb{R}_\xi)}$.

Remark 1.1. Actually, if $u_0(x)\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$, then the following decay estimate is established:

$$\|u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)}\le C(1+t)^{-\frac{1}{2}}, \tag{1.26}$$

where $C=C(\|u_0\|_{L^1(\mathbb{R})},\|u_0\|_{L^2(\mathbb{R})})$.

Exercise 1.1. Consider the following initial boundary value problem of the heat equation:

$$\begin{cases} u_t-cu_{xx}=0, & (t,x)\in(0,\infty)\times[0,2\pi],\\ u(0,x)=u_0(x), & x\in[0,2\pi],\\ u(t,0)=u(t,2\pi)=0, & t\in[0,\infty), \end{cases} \tag{1.27}$$

where $c>0$. Prove that $E_H(t):=\frac{1}{2}\|u(t,\cdot)\|^2_{L^2((0,2\pi)}$ decays exponential order.

Let $u(t,x)$ be a solution to the Cauchy problem of Schrödinger equation (1.17): If we introduce the energies of (1.17), and the corresponding Cauchy problem of $v(t,\xi)=\hat{u}(t,\xi)$ by

$$E_S(t)=\frac{1}{2}\int_{-\infty}^{\infty}|u(t,x)|^2dx, \quad \text{and} \quad \mathcal{E}_S(t,\xi)=\frac{1}{2}|v(t,\xi)|^2 \tag{1.28}$$

respectively, then we have the following energy conservation:

Proposition 1.6. *Let $u(t,x)$ be a solution of (1.17), and $u(t,x)\in C^1([0,\infty);L^\infty(\mathbb{R}_\xi))$, then the energy conservation laws $E_S(t)\equiv E_S(0)$ and $\mathcal{E}_S(t,\xi)\equiv\mathcal{E}_S(0,\xi)$ are established.*

Proof. (Exercise) □

Remark 1.2. If we define the energies to the solution of the Cauchy problem of the Klein-Gordon equation:

$$\begin{cases} u_{tt}-a^2u_{xx}+u=0, & (t,x)\in(0,\infty)\times\mathbb{R},\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), & x\in\mathbb{R}, \end{cases} \tag{1.29}$$

by

$$E_K(t)=\frac{1}{2}\int_{-\infty}^{\infty}\left(|u_t(t,x)|^2+a^2|u_x(t,x)|^2+|u(t,x)|^2\right)d\xi, \tag{1.30}$$

and

$$\mathcal{E}_K(t,\xi)=\frac{1}{2}\left(|v_t(t,\xi)|^2+a^2\xi^2|v(t,\xi)|^2+|v(t,\xi)|^2\right), \tag{1.31}$$

then the energy conservation laws: $E_K(t)\equiv E_K(0)$ and $\mathcal{E}_K(t,\xi)\equiv\mathcal{E}_K(0,\xi)$ are established.

2 Second order hyperbolic equations with constant coefficients

2.1 Hyperbolicity

Let us consider the following second order partial differential equation:

$$\left(\alpha\partial_t^2+2\beta\partial_t\partial_x+\gamma\partial_x^2\right)u=f(u,u_t,u_x), \tag{2.1}$$

where α,β,γ are real valued, and $\alpha\neq0$. Denoting $\partial_t=\lambda$ and $-\mathrm{i}\partial_x=\xi$, we have the following characteristic equation for the principal part of (2.1):

$$\alpha\lambda^2+2\mathrm{i}\beta\lambda\xi-\gamma\xi^2=0. \tag{2.2}$$

Then we classify the type of the equation (2.1) corresponding to the solutions of (2.2):

$$\lambda = \frac{-\mathrm{i}\beta\xi \pm \mathrm{i}\xi\sqrt{\beta^2 - \alpha\gamma}}{\alpha}$$

as follows:

$$\begin{cases} \beta^2 > \alpha\gamma \ \Leftrightarrow \ \text{hyperbolic;} \\ \beta^2 = \alpha\gamma \ \Leftrightarrow \ \text{parabolic;} \\ \beta^2 < \alpha\gamma \ \Leftrightarrow \ \text{elliptic.} \end{cases}$$

Let $\alpha = 1$. By partial Fourier transformation (2.1) is reduced to the following equation:

$$v_{tt} + 2\mathrm{i}\beta\xi v_t - \gamma\xi^2 v = 0. \tag{2.3}$$

REMARK 2.1. If $\beta = 0$, (2.1) is an elliptic equation for $\gamma > 0$. Then denoting $\gamma = c^2$ for $c \in \mathbb{R}$ we have

$$v(t,\xi) = C_1 \exp(c\xi t) + C_2 \exp(-c\xi t),$$

which gives that the solution growth in exponential order with respect to t since $\xi \neq 0$.

From now on we consider the hyperbolic case: $\gamma < 0$. Denoting $-\gamma = a^2$, $\beta = b$ and $c = \sqrt{a^2 + b^2}$ for $a > 0$, the solution to the Cauchy problem

$$v_{tt} + 2\mathrm{i}b\xi v_t + a^2\xi^2 v = 0, \quad v(0,\xi) = v_0(\xi), \quad v_t(0,\xi) = v_1(\xi) \tag{2.4}$$

is represented as follows:

$$v(t,\xi) = C_1 \exp(-\mathrm{i}(b-c)\xi t) + C_2 \exp(-\mathrm{i}(b+c)\xi t),$$

where

$$C_1 = \frac{b+c}{2c} v_0(\xi) + \frac{v_1(\xi)}{2\mathrm{i}c\xi}, \quad C_2 = -\frac{b-c}{2c} v_0(\xi) - \frac{v_1(\xi)}{2\mathrm{i}c\xi}.$$

Thus the structure of the solution is completely different between the elliptic equations and hyperbolic equations.

REMARK 2.2. Denoting $\lambda_{1\pm} = -\mathrm{i}\xi(b \pm c)$, we have

$$v_{tt} + 2\mathrm{i}b\xi v_t + a^2\xi^2 v = (\partial_t - \lambda_{1+})(\partial_t - \lambda_{1-})v = 0.$$

The solution to the equation $(\partial_t - \lambda_{1+})w = 0$ is given by $w = C_3(\xi)\exp(\lambda_{1+}t)$. Therefore, v is a solution to

$$(\partial_t - \lambda_{1-})v = C_3(\xi)\exp(\lambda_{1+}t).$$

Hence we have

$$v(t,\xi) = v_0(\xi)\mathrm{e}^{\lambda_{1-}t} + \mathrm{e}^{\lambda_{1-}t}\int_0^t C_3 \mathrm{e}^{\lambda_{1+}\tau - \lambda_{1-}\tau} d\tau = v_0(\xi)\mathrm{e}^{\lambda_{1-}t} + \frac{C_3}{\lambda_{1+} - \lambda_{1-}}\mathrm{e}^{\lambda_{1+}t}.$$

EXERCISE 2.1. Prove that if the equation $Pu = (\partial_t^2 + 2b\partial_x\partial_t - a^2\partial_x^2)u = 0$ is hyperbolic, then the solution to the Cauchy problem:

$$\begin{cases} Pu = 0, \ (t,x) \in (-\infty, 0] \cup [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1 \ x \in \mathbb{R} \end{cases} \tag{2.5}$$

is represented as follows:

$$u(t,x) = \frac{1}{2c}\left((b+c)u_0(x-(b-c)t) - (b-c)u_0(x-(b+c)t) + \int_{x-(b+c)t}^{x-(b-c)t} u_1(y)dy\right). \tag{2.6}$$

where $c = \sqrt{a^2 + b^2}$.

2.2 Reduction to first order system

Let us consider the following Cauchy problem:

$$\begin{cases} \left(\partial_t^2 + 2\mathrm{i}b\xi\partial_t + a^2\xi^2\right) v = 0, & (t,\xi) \in (0,\infty) \times \mathbb{R}, \\ v(0,x) = v_0(\xi), \quad v_t(0,\xi) = v_1(\xi), & \xi \in \mathbb{R}. \end{cases} \tag{2.7}$$

Then the the equation of (2.7) is represented as follows:

$$\left(\partial_t + \mathrm{i}\xi(b+c)\right)\left(\partial_t + \mathrm{i}\xi(b-c)\right) v = 0.$$

Here we remark that the operators $\partial_t + \mathrm{i}\xi(b+c)$ and $\partial_t + \mathrm{i}\xi(b-c)$ are commutative. Denoting

$$w_1 = v_t + \mathrm{i}\xi(b-c)v, \quad w_2 = v_t + \mathrm{i}\xi(b+c)v, \tag{2.8}$$

we have

$$\partial_t w_1 = -\mathrm{i}\xi(b+c)w_1, \quad \partial_t w_2 = -\mathrm{i}\xi(b-c)w_2.$$

Thus, the second order single equation of (2.7) is reduced to the following first order system:

$$\partial_t W = AW, \quad A(\xi) = \begin{pmatrix} -\mathrm{i}\xi(b+c) & 0 \\ 0 & -\mathrm{i}\xi(b-c) \end{pmatrix}, \quad W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{2.9}$$

Then the solution of (2.9) is represented as follows:

$$\begin{aligned} W(t,\xi) &= \exp\left(\int_{t_0}^t A(\xi)\,d\tau\right) W(t_0,\xi) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{t_0}^t A(\xi)\,d\tau\right)^k W(t_0,\xi) \\ &= \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\xi(b+c)(t-t_0)} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\xi(b-c)(t-t_0)} \end{pmatrix} W(t_0,\xi). \end{aligned}$$

It follows that

$$v_t(t,\xi) + \mathrm{i}\xi(b \pm c)v(t,\xi) = \mathrm{e}^{-\mathrm{i}\xi(b\pm c)(t-t_0)}\left(v_t(t_0,\xi) + \mathrm{i}\xi(b \pm c)v(t_0,\xi)\right).$$

Consequently, we have the following important property for hyperbolic model:

Theorem 2.1. *Let a, b be constants of real numbers. Then we have the following estimate of the energy conservation:*

$$|W(t,\xi)|^2 \equiv |W(t_0,\xi)|^2 \tag{2.10}$$

for any $t_0 \le t$, where

$$|W(t,\xi)|^2 = |v_t(t,\xi) - \mathrm{i}\xi(b+c)v(t,\xi)|^2 + |v_t(t,\xi) - \mathrm{i}\xi(b+c)v(t,\xi)|^2.$$

Proof.

$$\partial_t |W(t,\xi)|^2 = 2\Re\left\{\partial_t w_1 \overline{w_1}\right\} + 2\Re\left\{w_2 \overline{\partial_t w_2}\right\} = 0.$$

Hence we have $|W(t,\xi)|^2 \equiv |W(t_0,\xi)|^2$ for any $t \ge t_0$. □

Remark 2.3. The energy for the wave equation $\mathcal{E}_W(t,\xi)$ is not conserved for the general hyperbolic model (2.7), but $|W(t,\xi)|^2$ is conserved. Thus not $\mathcal{E}_W(t,\xi)$ but $|W(t,\xi)|^2$ should be a reasonable energy of microlocal version to the general hyperbolic equation (2.7).

The following lemma ensures the equivalence between $|W(t,\xi)|^2$ and $\mathcal{E}_W(t,\xi)$:

Lemma 2.1. *The relation $|W(t,\xi)|^2 \simeq |v_t(t,\xi)|^2 + \xi^2|v(t,\xi)|^2$ holds, where $f \simeq g$ denotes that there exists positive constants C_1 and C_2 such that the estimates $C_1 g \le f \le C_2 g$ uniformly hold.*

Proof. We can assume that $b \neq 0$, hence $c^2 > 0$. Let ε be a positive small constant. By Schwarz inequality we have

$$
\begin{aligned}
|W(t,\xi)|^2 =& 2|v_t|^2 + 2\xi^2\left(b^2+c^2\right)|v|^2 + 4\Re\left\{\mathrm{i}\xi b v_t \overline{v}\right\} \\
\geq & 2\varepsilon\left(|v_t|^2+\xi^2|v|^2\right) + 2(1-\varepsilon-\delta)|v_t|^2 + 2\xi^2\left(b^2+c^2-\varepsilon-\frac{b^2}{\delta}\right)|v|^2 \\
\geq & 2\varepsilon\left(|v_t|^2+\xi^2|v|^2\right),
\end{aligned}
$$

where δ, and ε are chosen small, and near 1 respectively, satisfying $0 < \delta, \varepsilon < 1$, $\varepsilon + \delta \leq 1$ and $c^2 \geq \varepsilon + b^2(\frac{1}{\delta} - 1)$. On the other hand, the estimate from above is trivial. □

3 Variable coefficients models

3.1 Background

The hyperbolic equation $(\partial_t^2 + 2b\partial_x\partial_t - a^2\partial_x^2)u = f(u, \partial_t u, \partial_x u)$ can be generalized to the following equation of variable coefficients:

$$
(\partial_t^2 + 2b(t,x)\partial_x\partial_t - a(t,x)^2\partial_x^2)u = f(t,x,\partial_t u, \partial_x u).
$$

Actually, the models of variable coefficients are natural generalizations from the point of view of mathematics. But they are important from the point of view for the applications of physics and engineering.

- Wave equation with variable propagation speed with respect to x:

$$
\left(\partial_t^2 - \partial_x a(x)^2\partial_x\right)u = 0: \quad \text{wave propagation in anisotropic media.} \tag{3.1}
$$

  Here we remark that the energy of this equation in $\mathbb{R}$ at t is given by

$$
E_W(t) = \frac{1}{2}\int_{-\infty}^{\infty}\left(a(x)^2|\partial_x u(t,x)|^2 + |\partial_t u(t,x)|^2\right)dx. \tag{3.2}
$$

  Then we have the energy conservation $E_W(t) \equiv E_W(0)$. Indeed, we have

$$
\begin{aligned}
\frac{d}{dt}E_W(t) =& \int_{-\infty}^{\infty}\left(a(x)^2\Re\{\partial_x u(t,x)\overline{\partial_x\partial_t u(t,x)}\} + \Re\{\partial_t^2 u(t,x)\overline{\partial_t u(t,x)}\}\right)dx \\
=& \lim_{R\to\infty}\left(a(R)^2\Re\{\partial_x u(t,R)\overline{\partial_t u(t,R)}\right) \\
& + \int_{-\infty}^{\infty}\left(-\Re\{\partial_x a(x)^2\partial_x u(t,x)\overline{\partial_t u(t,x)}\} + \Re\{\partial_t^2 u(t,x)\overline{\partial_t u(t,x)}\}\right)dx \\
=& 0.
\end{aligned}
$$

- Wave equation with time dependent speed:

$$
\left(\partial_t^2 - a(t)^2\partial_x^2\right)u = 0 \tag{3.3}
$$

  is a linearized model of non-linear wave equation of Kirchhoff type:

$$
\partial_t^2 u - \left(1 + \int_{-\infty}^{\infty}|\partial_x u(t,x)|^2 dx\right)\partial_x^2 u = 0. \tag{3.4}
$$

  Here the Kirchhoff equation has the following energy conservation law:

$$
E_{Ki}(t) := \frac{1}{2}\left(\|\partial_t u(t,\cdot)\|^2 + \int_0^{\|\partial u(t,\cdot)\|^2}(1+\eta)d\eta\right). \tag{3.5}
$$

Indeed, we see that

$$\frac{d}{dt}E_{Ki}(t) = \left(\Re\left(\partial_{tt}u, \partial_t u\right)_{L^2(\mathbb{R})} + \left(1 + \|\partial u(t,\cdot)\|^2\right)\Re\left(\partial\partial_t u, \partial u\right)_{L^2(\mathbb{R})}\right) = 0.$$

The global solvability for Kirchhoff equation is said that a very hard open problem. Global solvability is solved essentially for realanalytic, and small data in [1], and [6] respectively. [17] is helpful to understand the overview for the research of Kirchhoff equation.

- The Cauchy problems of hyperbolic equations with constant coefficients are L^2 wellposed, that is, the following energy estimate is established:

$$\|u(t,\cdot)\|_{H^1(\mathbb{R})} + \|\partial_t u(t,\cdot)\|_{L^2(\mathbb{R})} \le C\left(\|u_0(\cdot)\|_{H^1(\mathbb{R})} + \|u_1(\cdot)\|_{L^2(\mathbb{R})}\right) \tag{3.6}$$

for any $t \in [0,T]$, where C may depend on T. However, the L^2 well-posedness is not true in general for the hyperbolic equations with variable coefficients. If the coefficients are t dependent, then the solutions of the characteristic equation $\lambda^2 + 2ib(t)\xi\lambda + a(t)\xi^2 = 0$ can coincide for $\xi \neq 0$; this brings a crucial problem. Such equations are called weakly hyperbolic equations, and there are number of studies about it.

Let p and q be positive real numbers. The following weakly hyperbolic equation is studied by [13]:

$$\left(\partial_t^2 - t^{2p}\partial_x^2 + t^{q-1}\partial_x\right)u = 0. \tag{3.7}$$

When $t \to 0$, then the term of ∂_x^2 cannot be dominant the lower order term anymore by the size of p to q. Indeed, the Cauchy problem of (3.7) is C^∞ well-posed near $t = 0$ if and only if $q \ge p$. This means that the singular effect of the coefficient; the degeneration of as $t \to 0$, brings loss of derivative of the solution.

Let $a(t)$ is bounded from above and below by positive constants. The solution to the wave equation with time dependent propagation speed:

$$\left(\partial_t^2 - a(t)^2\partial_x\right)u = 0 \tag{3.8}$$

is possible to lose its regularity without degeneration of the principal part and lower order terms. A natural energy of (3.8) will be $E_W(t) = \frac{1}{2}(a(t)^2\|\partial u(t,\cdot)\|^2_{L(\mathbb{R})^2} + \|\partial_t u(t,\cdot)\|^2_{L^2(\mathbb{R})}$ on the analogy of the constant coefficient case. However, we immediately meet a crucial problem if we estimate $E_W(t)$ to reduce a differential inequality and Gronwall's inequality; this is not only a technical but also an essential problem. Indeed, the L^2 well-posedness is not true in general if $a(t)$ is not a Lipchitz continuous function. The influence of singularities of the propagation speed $a(t)$, in the sense of non-Lipschitz continuity, to the order of loss of regularity of the solution is studied in [4] and [5], for instance.

- Let $b(t,x) \ge 0$. The dissipative wave equation

$$\left(\partial_t^2 - \partial_x^2 + 2b(t,x)\partial_t\right)u = 0, \tag{3.9}$$

describes the wave phenomenon with variable friction $b(t,x)$. We immediately see that $\frac{d}{dt}E_W(t) \le 0$, hence the total energy is decreasing. The next problems are the energy decay and the decay order. If b is a positive constant, then the following decay estimate is established:

$$E_W(t) = C(1+t)^{-1},$$

where the constant C depends on $\|u_0\|_{H^1}$ and $\|u_1\|_{L^2}$. On the other hand, if $b = b(x) = \mathcal{O}(1+|x|)^{-\alpha}$, or $b = b(t) = \mathcal{O}(1+t)^{-\alpha}$ for $\alpha > 1$, then the energy does not decay. Indeed, if $b(t) = (1+t)^{-\alpha}$, then we have

$$\partial_t E_W(t) = -2b(t)\|u_t\|^2_{L^2} \ge -4b(t)E(t),$$

it follows that

$$E(t) \geq E(0) \exp\left(-4\int_0^t (1+s)^{-\alpha} ds\right) \geq CE(0).$$

If $b(t) = b_0(1+t)^{-1}$ and $b_0 > 0$, then we have

$$E(t) \leq C(1+t)^{-\sigma}, \quad \sigma = 2\min\{b_0, 1\}.$$

Generally, one may expect that stronger dissipation brings faster decay estimate, however, it is not true. For the energy decay and non-decay problems are studied in [14], [15], [19], [20] and references of them.

3.2 Factorized model

Let us consider the following hyperbolic equation with lower order term:

$$\left(\partial_t^2 - a(t)^2\partial_x^2 - a'(t)\partial_x\right) u = 0. \tag{3.10}$$

By partial Fourier transformation we have

$$v_{tt} + a(t)^2\xi^2 v - \mathrm{i}a'(t)\xi v = 0. \tag{3.11}$$

Here this equation can be factorized as follows:

$$\left(\partial_t - \mathrm{i}\xi a(t)\right)\left(\partial_t + \mathrm{i}\xi a(t)\right) v = 0. \tag{3.12}$$

Therefore, the solution of (3.11) with the initial data $(v(0,\xi), v_t(0,\xi)) = (v_0(\xi), v_1(\xi))$ is represented as follow:

$$\begin{aligned} v(t,\xi) =& v_0(\xi)\exp\left(-\mathrm{i}\xi\int_0^t a(\tau)d\tau\right) \\ &+ \left(\frac{v_1(\xi)}{-\mathrm{i}\xi a(0)} - v_0(\xi)\right)\exp\left(-\mathrm{i}\xi\int_0^t a(\tau)d\tau\right)\int_0^t \exp\left(2\mathrm{i}\xi\int_0^\tau a(\sigma)d\sigma\right) d\tau. \end{aligned}$$

Let w be a solution of $w_t - \mathrm{i}\xi a(t)w = 0$. Then v is a solution to the following inhomogeneous equation:

$$\left(\partial_t + \mathrm{i}\xi a(t)\right) v = w, \tag{3.13}$$

it follows that we have the following system:

$$\partial_t Y = \begin{pmatrix} \mathrm{i}\xi a(t) & 0 \\ 1 & -\mathrm{i}\xi a(t) \end{pmatrix} Y, \quad Y = \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} v_t + \mathrm{i}\xi a(t)v \\ v \end{pmatrix}. \tag{3.14}$$

Therefore, the following property for a sort of conservation is established:

$$|v_t(t,\xi) + \mathrm{i}\xi a(t)v(t,\xi)|^2 = |v_1(\xi) + \mathrm{i}\xi a(0)v_0(\xi)|^2, \quad \forall(t,\xi). \tag{3.15}$$

Generally, the factorized hyperbolic equations with time dependent coefficients are given as follows:

$$\begin{aligned} &\left(\partial_t^2 - \lambda_1(t)\lambda_2(t)\xi^2 v - \mathrm{i}\xi\left(\lambda_1(t) + \lambda_2(t)\right)\partial_t - \mathrm{i}\lambda_2'(t)\xi\right) v = 0 \\ \Leftrightarrow\ &\left(\partial_t - \mathrm{i}\xi\lambda_1(t)\right)\left(\partial_t - \mathrm{i}\xi\lambda_2(t)\right) v = 0. \end{aligned}$$

Then the solution is represented by

$$\begin{aligned} v(t,\xi) =& v_0(\xi)\exp\left(\mathrm{i}\xi\int_0^t \lambda_2(\tau)d\tau\right) \\ &+ \left(\frac{v_1(\xi)}{\mathrm{i}\xi\lambda_2(0)} - v_0(\xi)\right)\exp\left(\mathrm{i}\xi\int_0^t \lambda_2(\tau)d\tau\right)\int_0^t \exp\left(\mathrm{i}\xi\int_0^\tau \left(\lambda_1(\sigma) - \lambda_2(\sigma)\right)d\sigma\right) d\tau. \end{aligned}$$

We have observed from the examples above that the factorized models are essentially single equations. Thus we should focus non-factorized model from now on.

3.3 Reduction to first order systems

Let us consider the following Cauchy problem of the wave equation with variable propagation speed:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\partial_x^2\right) u = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}. \end{cases} \tag{3.16}$$

By partial Fourier transformation the equation is rewritten to the following problem:

$$\begin{cases} \left(\partial_t^2 + a(t)^2\xi^2\right) v = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}, \\ v(0,\xi) = v_0(\xi), \quad v_t(0,\xi) = v_1(\xi), \quad x \in \mathbb{R}, \end{cases} \tag{3.17}$$

where $a \in C^1([0,\infty))$ and $0 < a_0 \le a(t) \le a_1$. Then the following energy is naturally proposed on an analogy of the constant coefficient model:

$$\mathcal{E}(t,\xi) = \frac{1}{2}\left(a(t)^2\xi^2|v(t,\xi)|^2 + |v_t(t,\xi)|^2\right). \tag{3.18}$$

Here the total energy of (3.16) is given by

$$E(t) = \frac{1}{2}\left(a(t)^2\|\partial u(t,\cdot)\|^2 + \|u_t(t,\cdot)\|^2\right). \tag{3.19}$$

Differentiating $\mathcal{E}(t,\xi)$ with respect to t we have

$$\partial_t\mathcal{E}(t,\xi) = a'(t)a(t)\xi^2|v(t,\xi)|^2.$$

Here we observe that if $a'(t)$ is not a constant, then the energy $\mathcal{E}(t,\xi)$ is not conserved. Generally, we cannot expect the energy conservation for variable coefficient model, because this effect describes an external force. (We don't say that any conserved quantity does not exist.) Therefore, we introduce a sort of conservation, which is called the generalized energy conservation in [8] by

$$E(t) \simeq E(0), \tag{3.20}$$

and

$$\mathcal{E}(t,\xi) \simeq \mathcal{E}(0,\xi). \tag{3.21}$$

Here we note that these properties do not require that $\lim_{t\to\infty}|E(t)-E(0)| = 0$ and $\lim_{t\to\infty}|\mathcal{E}(t,\xi) - \mathcal{E}(0,\xi)| = 0$. In this sense, (3.20) and (3.21) are not perturbation of the constant coefficient model. We immediately have (3.20) from (3.21), hence we shall consider only the latter estimate from now on.

If $a(t)$ is a monotone increasing function, then we have

$$-\frac{2a'(t)}{a(t)}\mathcal{E}(t,\xi) \le \partial_t\mathcal{E}(t,\xi) \le \frac{2a'(t)}{a(t)}\mathcal{E}(t,\xi).$$

Therefore, by Gronwall's inequality we have

$$\frac{a(0)^2}{a(t)^2}\mathcal{E}(0,\xi) \le \mathcal{E}(t,\xi) \le \frac{a(t)^2}{a(0)^2}\mathcal{E}(0,\xi),$$

that is, (3.21). If $a(t)$ is monotone decreasing, then we have (3.21) by the same argument. If $a'(t)$ changes its sign at most finite time, then we also have (3.21) immediately. Therefore, our problem must be restricted in the case that $a'(t)$ changes its sign infinitely many times.

If $a'(t)$ changes its sign infinite times, but $a'(t) \in L^1((0,\infty))$, then we easily see that the following proposition is valid:

Proposition 3.1. *If $a'(t) \in L^1((0,\infty)$, then (3.21) holds.*

Let us reduce the equation of (3.17) to the following first order system:

$$\partial_t V_1 = A_1 V_1, \quad V_1 = \begin{pmatrix} v_t - \mathrm{i}\xi a v \\ v_t + \mathrm{i}\xi a v \end{pmatrix}, \tag{3.22}$$

where

$$A_1 = \begin{pmatrix} -\mathrm{i}\xi a + \frac{a'}{2a} & -\frac{a'}{2a} \\ -\frac{a'}{2a} & \mathrm{i}\xi a + \frac{a'}{2a} \end{pmatrix} = \Lambda_1 + B_1 = \Phi_1 + R_1, \tag{3.23}$$

$$\Lambda_1 = \begin{pmatrix} -\mathrm{i}\xi a & 0 \\ 0 & \mathrm{i}\xi a \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{a'}{2a} & -\frac{a'}{2a} \\ -\frac{a'}{2a} & \frac{a'}{2a} \end{pmatrix},$$

$$\Phi_1 = \begin{pmatrix} -\mathrm{i}\xi a + \frac{a'}{2a} & 0 \\ 0 & \mathrm{i}\xi a + \frac{a'}{2a} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & -\frac{a'}{2a} \\ -\frac{a'}{2a} & 0 \end{pmatrix}.$$

Here we note that the decomposition $A_1 = \Lambda_1 + B_1$ is natural from the point of view for the perturbation of constant coefficient model. On the other hand, we prefer to employ the other decomposition, which the diagonal entries are not pure imaginary valued. Indeed, such decomposition is essential in our consideration for the future.

Let us denote

$$\phi_1 = -\mathrm{i}\xi a + \frac{a'}{2a} \quad \text{and} \quad r_1 = -\frac{a'}{2a}.$$

Then we have

$$\Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & r_1 \\ \overline{r_1} & 0 \end{pmatrix}.$$

Let $t_0 \in [0, t)$, and define $\Xi_1 = \Xi_1(t_0, t, \xi)$ by

$$\begin{aligned} \Xi_1(t_0, t, \xi) &= \begin{pmatrix} \exp\left(\int_{t_0}^t \phi_1(\tau, \xi) d\tau\right) & 0 \\ 0 & \exp\left(\int_{t_0}^t \overline{\phi_1(\tau, \xi)} d\tau\right) \end{pmatrix} \\ &= \sqrt{\frac{a(t)}{a(t_0)}} \begin{pmatrix} \exp\left(\mathrm{i}\int_{t_0}^t \phi_{1\Im}(\tau, \xi) d\tau\right) & 0 \\ 0 & \exp\left(-\mathrm{i}\int_{t_0}^t \phi_{1\Im}(\tau, \xi) d\tau\right) \end{pmatrix}, \end{aligned}$$

where $\phi_{1\Im} = \Im\{\phi_1\}$. We also denote $\phi_{1\Re} = \Re\{\phi_1\}$. Then we have

$$\partial_t Y_1 = \tilde{R}_1 Y_1, \quad Y_1 = \Xi_1^{-1} V_1, \quad \tilde{R}_1 = \Xi_1^{-1} R_1 \Xi_1. \tag{3.24}$$

Here we note that Ξ_1 and Ξ_1^{-1} are uniformly bounded from $\mathbb{R}^2$ to $\mathbb{R}^2$ in $[t_0, t] \times \mathbb{R}$. Hence we have

$$|Y_1(t, \xi)|^2 \simeq |V_1(t, \xi)|^2 \simeq \mathcal{E}(t, \xi) \quad \text{and} \quad \|\tilde{R}_1\| \simeq \|R_1\|. \tag{3.25}$$

Therefore, we have

$$\partial_t |Y_1|^2 = 2\Re\left(\partial_t Y_1, Y_1\right)_{\mathbb{C}^2} = 2\Re\left(\partial_t Y_1, Y_1\right)_{\mathbb{C}^2} \le C|r_1||Y_1|.$$

Consequently, if $r_1 \in L^1((t_0, t))$, then we obtain

$$\mathcal{E}(t, \xi) \simeq \mathcal{E}(t_0, \xi). \tag{3.26}$$

The consideration above can be applied to conclude the following proposition without any difficulties:

Proposition 3.2. *Let $0 \le t_0 < T$ and $\Omega \subset \mathbb{R}_\xi$, where $T = \infty$ is admissible. Let us consider the following first order system:*

$$\partial_t V = AV, \quad A(t,\xi) = \begin{pmatrix} \phi_+(t,\xi) & r_+(t,\xi) \\ r_-(t,\xi) & \phi_-(t,\xi) \end{pmatrix} \tag{3.27}$$

in $[t_0, T) \times \Omega$. If there exists a positive constant C such that

$$\sup_{t>t_0, \xi\in\Omega} \left\{ \left| \int_{t_0}^{t} \phi_{\Re\pm}(\tau,\xi) d\tau \right| \right\} < C, \tag{3.28}$$

and

$$\sup_{\xi\in\Omega} \left\{ \int_{t_0}^{T} |r_\pm(\tau,\xi)| d\tau \right\} < \infty, \tag{3.29}$$

then we have

$$|V(s,\xi)| \simeq |V(t,\xi)| \tag{3.30}$$

uniformly in $s, t \in [t_0, T)$ and $\xi \in \Omega$.

Exercise 3.1. Prove Proposition 3.2.

A typical example of $a(t)$ from a conclusion of Proposition 3.2 for (3.22) is the following:

$$a(t) = 2 + \cos\left((1+t)^{1-\beta}\right), \quad \beta > 1.$$

However, it requires to be $\beta > 1$ for the condition (3.29), and this is not an interesting case. r_1 describes the order of the oscillating speed of $a(t)$, and faster oscillation, that is, smaller β, will give a worse influence to the stability of the energy. From now on we consider the following example, which is a limit case as $\beta \to 1$:

$$a(t) = 2 + \cos\left(\log(1+t)\right).$$

We see that $a'(t) \notin L^1((0,\infty))$, hence (3.21) is no really clear.

3.4 C^2 property of the coefficient for wave type equations

Let us consider the wave type equation (3.17). Then the following theorem can be proved:

Theorem 3.1 ([16])**.** *Let $a(t) \in C^2([0,\infty))$ satisfy $0 < a_0 \le a(t) \le a_1$ and*

$$\left| a^{(k)}(t) \right| \le C_k (1+t)^{-k} \quad (k = 1, 2), \tag{3.31}$$

then (3.21) is established.

Firstly, we consider the energy estimate in low frequency part. Let a_∞ be a positive constant. We define the energy $\mathcal{E}_\infty(t,\xi)$ by

$$\mathcal{E}_\infty(t,\xi) = \frac{1}{2}\left(a_\infty^2 \xi^2 |v(t,\xi)|^2 + |v_t(t,\xi)|^2 \right). \tag{3.32}$$

Then we have

$$\partial_t \mathcal{E}_\infty(t,\xi) = \left(a_\infty^2 - a(t)^2\right) \xi^2 \Re\{v \overline{v_t}\} \le \frac{\left|a_\infty^2 - a(t)^2\right| |\xi|}{a_\infty} \mathcal{E}_\infty(t,\xi) \le C|\xi| \mathcal{E}_\infty(t,\xi).$$

It follows that

$$\mathcal{E}_\infty(t,\xi) \le \exp\left(C(1+t)|\xi|\right) \mathcal{E}_\infty(0,\xi). \tag{3.33}$$

Here we note that $\mathcal{E}_\infty(t,\xi) \simeq \mathcal{E}(t,\xi)$ uniformly in the phase space $\{(t,\xi) \in [0,\infty) \times \mathbb{R}\}$. This estimate is too rough if we need the energy estimate in the whole space. But the estimate (3.21) is true in the following restricted aria; we shall denote this aria by Z_Ψ:

$$Z_\Psi = \{(t,\xi)\,;\ (1+t)|\xi| \le N\}\,, \tag{3.34}$$

where N is a positive constant. Indeed, N is possible to be chosen any large size, and we will suppose that N is large enough. Consequently, we must consider only in the following zone; we shall denote this aria by Z_H:

$$Z_H = \{(t,\xi)\,;\ (1+t)|\xi| \ge N\}\,. \tag{3.35}$$

Here we denote the curve, which separates Z_Ψ and Z_H by t_ξ. Only the difference from the constant coefficient case is whether $B_1 = 0$ or not. If we don't separate the phase space, one cannot compare the order of Λ_1 and B_1. However, in Z_H we see that Λ_1 is dominant. Indeed, we see that

$$\|B_1\| = |r_1(t,\xi)| \le C(1+t)^{-1} \le CN^{-1}|\xi| \simeq a(t)|\xi| = \|\Lambda_1\|.$$

Let us introduce the following symbol classes in order to discuss the order of the coefficients systematically. Let m be a non-negative integers, k and l be integers. The symbol class $S^{(m)}\{k,l\}$ is the class of functions $f(t,\xi)$ which satisfy

$$\left|\partial_t^j f(t,\xi)\right| \le C_j|\xi|^k(1+t)^{-l-j} \quad (j=0,\cdots,m) \text{ in } (t,\xi) \in Z_H.$$

Indeed, we see that $\|\Lambda_1\| = a(t)|\xi| \in S^{(2)}\{1,0\}$, $\|B_1\| = \|R_1\| = |r_1| \in S^{(1)}\{0,1\}$. For the symbol classes tha following properties are valid:

(i) Let $k_1+l_1 = k_2+l_2$. If $f_1 \in S^{(m_1)}\{k_1,l_1\}$ and $f_2 \in S^{(m_2)}\{k_2,l_2\}$, then we have $f_1+f_2 \in S^{(m)}\{k,l\}$, where $m = \min\{m_1,m_2\}$, $k = \max\{k_1,k_2\}$ and $l = \min\{l_1,l_2\}$.

(ii) If $f_1 \in S^{(m_1)}\{k_1,l_1\}$ and $f_2 \in S^{(m_2)}\{k_2,l_2\}$, then we have $f_1f_2 \in S^{(\min\{m_1,m_2\})}\{k_1+k_2,l_1+l_2\}$.

(iii) If $f \in S^{(m)}\{k,l\}$, then $\partial_t f \in S^{(m-1)}\{k,l+1\}$.

(iv) If $f \in S^{(m)}\{k,l\}$, then $f \in S^{(m)}\{k+1,l-1\}$. Moreover, if $f \in S^{(m)}\{-k,k\}$ with $k > 0$, then we have $|f| \ll 1$ since N large.

(v) If $f(t,\xi) \in S^{(0)}\{-1,2\}$, then we have $\int_{t_\xi}^t |f(\tau,\xi)|d\tau$ is uniformly bounded in Z_H.

Thus the principal part of A_1 is $\phi_{1\Im} \in S^{(2)}\{1,0\}$, which is the imaginary part of the diagonal entry; on the other hand, $\phi_{1\Re}, r_1 \in S^{(1)}\{0,1\}$ are subprincipal part. However, $r_1 \in S^{(1)}\{0,1\}$ is not enough small order to conclude (3.21).

Remark 3.1. We distinguish by the following three levels for the influence of the symbols in A:

- $\phi_{\Im\pm}$: non influence to the energy estimate;
- $\phi_{\Re\pm}$: influence as the amplitude of the energy after (Riemann) integration.
- $r_\pm$: cannot be derived any properties even if the real part and the imaginary part are separated. Consequently, we can observe as the order of the amplitude of the energy after L^1 type integration.

Therefore, some diagonalization steps of A are very important to derive precise estimates for the energy.

Let us carry out the next step of diagonalization procedure to A_1. The eigenvalues $\lambda_{1\pm}$ of A_1 and their corresponding eigenvectors $(1,\theta_{1+})^T$, $(\theta_{1-},1)^T$ are given by

$$\lambda_{1\pm} = \frac{2\phi_{1\Re} \pm \mathrm{i}\sqrt{|\phi_{1\Im}|^2 - 4|r_1|^2}}{2},$$

and

$$\theta_{1+} = \frac{\lambda_{1+} - \phi_1}{r_1}, \quad \theta_{1-} = \frac{\lambda_{1-} - \overline{\phi_1}}{\overline{r_1}}.$$

Hence we have

$$\Theta_1^{-1} A_1 \Theta_1 = \begin{pmatrix} \lambda_{1+} & 0 \\ 0 & \lambda_{1-} \end{pmatrix}, \quad \Theta_1 = \begin{pmatrix} 1 & \theta_{1-} \\ \theta_{1+} & 1 \end{pmatrix}.$$

Here we introduce the following lemma:

Lemma 3.1. *We have the followings:*

(i) $\lambda_1 = \lambda_{1+} = \overline{\lambda_{1-}}$*, hence* $\theta_1 = \theta_{1+} = \overline{\theta_{1-}}$*.*

(ii) $\lambda_1 - \phi_1 \in S^{(1)}\{-1,2\}$ *and* $\theta_1 \in S^{(1)}\{-1,1\}$*, it follows that* Θ_1 *is invertible in* Z_H*.*

The proof of this lemma will be discussed later as a special case of general second order model.

If Θ_1 is independent of t, then we immediately have (3.21) by using this lemma. However, ∂_t and Θ_1 is not commutative, hence (3.22) cannot be a diagonal system by this diagonalization procedure. Indeed, we have

$$\Theta_1^{-1} \partial_t \Theta_1 = \frac{1}{1-|\theta_1|^2} \begin{pmatrix} 1 & -\overline{\theta_1} \\ -\theta_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & \overline{\theta_1}' \\ \theta_1' & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\overline{\theta_1}\theta_1'}{1-|\theta_1|^2} & \frac{\overline{\theta_1}'}{1-|\theta_1|^2} \\ \frac{\theta_1'}{1-|\theta_1|^2} & -\frac{\theta_1 \overline{\theta_1}'}{1-|\theta_1|^2} \end{pmatrix}.$$

If we denote

$$r_2 = \frac{\overline{\theta_1}'}{1-|\theta_1|^2}, \quad \phi_2 = \lambda_1 - \frac{\overline{\theta_1}\theta_1'}{1-|\theta_1|^2},$$

then we have

$$\partial_t V_2 = (\Phi_2 + R_2) V_2, \quad V_2 = \Theta_1^{-1} V_1, \quad \Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & r_2 \\ \overline{r_2} & 0 \end{pmatrix}. \tag{3.36}$$

Here we note the following facts:

Lemma 3.2. *(i)* Θ_1 *and* Θ_1^{-1} *are bounded from* $\mathbb{C}^2$ *to* $\mathbb{C}^2$ *uniformly in* Z_H*.*

(ii) $\phi_{2\Re} - \phi_{1\Re} \in S^{(0)}\{-2,3\} \subset S^{(0)}\{-1,2\}$ *and* $r_2 \in S^{(0)}\{-1,2\}$*. It follows that*

$$|\phi_{2\Re} - \phi_{1\Re}| \le C|\xi|^{-1}(1+t)^{-2} \ \ \text{and} \ \ |r_2| \le C|\xi|^{-1}(1+t)^{-2}. \tag{3.37}$$

Hence we can apply Proposition 3.2 in Z_H *for* $A = \Phi_2 + R_2$*.*

By this lemma conclude the proof of Theorem 3.1.

3.5 C^2 property of the coefficient for general hyperbolic equations

Let $t_0 \ge 0$ be a fixed initial time. We consider the following Cauchy problem of a homogeneous hyperbolic equation with variable coefficients:

$$\begin{cases} \left(\partial_t^2 + 2b(t)\partial_t\partial_x + a(t)^2\partial_x^2\right) u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(t_0,x) = u_0(x), \quad u_t(t_0,x) = u_1(x), & x \in \mathbb{R}. \end{cases} \tag{3.38}$$

By partial Fourier transformation the equation is rewritten to the following problem:

$$\begin{cases} \left(\partial_t^2 + 2ib(t)\xi + a(t)^2\xi^2\right) v = 0, & (t,\xi) \in (t_0,\infty) \times \mathbb{R}, \\ v(t_0,\xi) = v_0(\xi), \quad v_t(t_0,\xi) = v_1(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{3.39}$$

where $a(t), b(t) \in C^2([t_0,\infty))$ and $0 < c_0 = \sqrt{a(t)^2 + b(t)^2} =: c(t) \le c_1$.

Our next problem is to derive the same property as Theorem 3.1. It may be a natural expectation, but it is not true in general. Indeed we have the following theorem:

Theorem 3.2. *If $a(t)$ and $b(t)$ satisfy*

$$\left|a^{(k)}(t)\right| + \left|b^{(k)}(t)\right| \le C_k(1+t)^{-k} \quad (k=1,2), \tag{3.40}$$

and

$$\sup_{t>0}\left\{\left|\int_0^t \frac{b'(\tau)}{\sqrt{a(\tau)^2+b(\tau)^2}}d\tau\right|\right\} < \infty, \tag{3.41}$$

then the following energy estimate is valid uniformly with respect to $(t,\xi) \in [t_0,\infty)\times\mathbb{R}$:

$$\mathcal{E}(t,\xi) \simeq \mathcal{E}(t_0,\xi). \tag{3.42}$$

Here the assumption (3.41) is crucial for the estimate (3.42). Indeed we also have the following theorem:

Theorem 3.3. *Let $a(t)$ and $b(t)$ satisfy (3.40). If there exist a positive constant C_0 and a sequence of intervals $I_j = [s_j, t_j] \subset [0,\infty)$ $(j = 1,2,\cdots)$ such that*

$$\lim_{j\to\infty}\int_{I_j} \frac{b'(\tau)}{\sqrt{a(\tau)^2+b(\tau)^2}}d\tau = \infty, \tag{3.43}$$

and

$$\inf_{s_j<s<t<t_j}\left\{\int_s^t \frac{b'(\tau)}{\sqrt{a(\tau)^2+b(\tau)^2}}d\tau\right\} \ge -C_0, \tag{3.44}$$

then there exists initial data of (3.39) at $t_0 = s_j$ such that

$$\mathcal{E}(s_j,\xi) \le 1 \quad \text{and} \quad \lim_{j\to\infty}\mathcal{E}(t_j,\xi) = \infty. \tag{3.45}$$

Remark 3.2. Theorem 3.3 is a special case in the result of [12].

Remark 3.3. Theorem 3.2 and Theorem 3.3 are considered from a point of view for the L^2 well-posedness for non-Lipschitz coefficients in [3], [9] (for weakly hyperbolic models in [11]), and [7].

From now on we shall prove Theorem 3.2 under too precise consideration, which is not necessary if one only prove it by C^2 property of the coefficients. However, we will discuss not only C^2 but also C^m property of the coefficients, and then our precise consideration will be necessary.

Proof of Theorem 3.2. In Z_Ψ we have

$$\partial_t\mathcal{E}_\infty(t,\xi) = \left(a_\infty^2 - a(t)^2\right)\xi^2\Re\{v\overline{v_t}\} \le C|\xi|\mathcal{E}_\infty(t,\xi),$$

it follows that $\mathcal{E}_\infty(t,\xi) \le C\mathcal{E}_\infty(0,\xi)$ in Z_Ψ.

In Z_H let us start from the following system:

$$\partial_t V_1 = (\Phi_1 + R_1)V_1, \quad V_1 = \begin{pmatrix} v_t + \mathrm{i}\xi(b-c)v \\ v_t + \mathrm{i}\xi(b+c)v \end{pmatrix}, \tag{3.46}$$

where

$$\phi_{1\pm} = -\mathrm{i}\xi(b\pm c) \mp \frac{b'}{2c} + \frac{c'}{2c}, \quad r_{1\pm} = \pm\frac{b'}{2c} - \frac{c'}{2c}. \tag{3.47}$$

The eigenvalues $\lambda_{1\pm}$ of $\Phi_1 + R_1$, and their corresponding eigenvectors $(1,\theta_{1+})^T$, $(\theta_{1-},1)^T$ are given by

$$\lambda_{1\pm} = \frac{\phi_{1+}+\phi_{1-} \pm \sqrt{(\phi_{1+}-\phi_{1-})^2 + 4r_{1+}r_{1-}}}{2}, \quad \theta_{1\pm} = \frac{\lambda_{1\pm}-\phi_{1\pm}}{r_{1\pm}}.$$

Then we have the following lemma:

Lemma 3.3. $\frac{1}{c} \in S^{(2)}\{0,0\}$ *and* $\theta_\pm \in S^{(1)}\{-1,1\}$. *It follows that Θ_1 is invertible in Z_H.*

Proof. We observe that

$$r_{1\pm} = \pm\frac{b'}{2c} - \frac{c'}{2c} \in S^{(1)}\{0,1\}, \quad \frac{1}{\phi_{1+}-\phi_{1-}} = \frac{1}{-2i\xi c - \frac{b'}{c}} \in S^{(1)}\{-1,0\}.$$

Indeed, noting $\frac{b'}{c} \in S^{(1)}\{0,1\}$, we see that

$$\left|-2i\xi c - \frac{b'}{c}\right| \geq 2c|\xi| - \frac{|b'|}{c} \geq 2c|\xi| - C(1+t)^{-1} \geq c|\xi|.$$

Hence we have $\frac{4r_{1+}r_{1-}}{(\phi_{1+}-\phi_{1-})^2} \in S^{(1)}\{-2,2\}$. By the approximation

$$\sqrt{1+\delta} = 1 + \frac{1}{2}\delta + q(\delta)\delta^2, \quad q(\delta) = \mathcal{O}(1),$$

we have the following representations:

$$\begin{aligned}
\lambda_{1\pm} =& \frac{\phi_{1+}+\phi_{1-}}{2} \pm \frac{\phi_{1+}-\phi_{1-}}{2}\sqrt{1+\frac{4r_{1+}r_{1-}}{(\phi_{1+}-\phi_{1-})^2}} \\
=& \frac{\phi_{1+}+\phi_{1-}}{2} \pm \frac{\phi_{1+}-\phi_{1-}}{2}\left(1+\frac{2r_{1+}r_{1-}}{(\phi_{1+}-\phi_{1-})^2} + \frac{2q_0 r_{1+}^2 r_{1-}^2}{(\phi_{1+}-\phi_{1-})^4}\right) \\
=& \phi_{1\pm} \pm \left(\frac{r_{1+}r_{1-}}{\phi_{1+}-\phi_{1-}} + \frac{q_0 r_{1+}^2 r_{1-}^2}{(\phi_{1+}-\phi_{1-})^3}\right),
\end{aligned}$$

and

$$\theta_{1\pm} = \frac{-\phi_{1\pm}+\lambda_{1\pm}}{r_{1\pm}} = \pm\left(\frac{r_{1\mp}}{\phi_{1+}-\phi_{1-}} + \frac{q_0 r_{1+}r_{1-}r_{1\mp}}{(\phi_{1+}-\phi_{1-})^3}\right) = \pm r_{1\mp}\left(\frac{1}{\phi_{1+}-\phi_{1-}} + \frac{q_0 r_{1+}r_{1-}}{(\phi_{1+}-\phi_{1-})^3}\right),$$

where $q_0 \in S^{(1)}\{0,0\}$ is represented by

$$q_0 = \sum_{j=0}^{\infty} \alpha_j \left(\frac{r_{1+}r_{1-}}{(\phi_{1+}-\phi_{1-})^2}\right)^j$$

with real sequence $\{\alpha_j\}$ satisfying $\sum_{j=0}^{\infty}|\alpha_l| < \infty$. Therefore, we have $\theta_{1\pm} \in S^{(1)}\{-1,1\}$. □

By the diagonalizer Θ_1 of $\Phi_1 + R_1$, which is bounded from $\mathbb{C}^2$ to $\mathbb{C}^2$ in Z_H, we have

$$\partial_t V_2 = (\Phi_2 + R_2)V_2, \quad V_2 = \Theta_1^{-1}V_1, \quad \Phi_2 = \begin{pmatrix} \phi_{2+} & 0 \\ 0 & \phi_{2-} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & r_{2+} \\ r_{2-} & 0 \end{pmatrix}, \tag{3.48}$$

where

$$r_{2\pm} = -\frac{\theta'_{1\mp}}{1-\theta_{1+}\theta_{1-}} \in S^{(0)}\{-1,2\},$$

and

$$\phi_{2\pm} = \lambda_{1\pm} + \frac{\theta_{1\mp}\theta'_{1\pm}}{1-\theta_{1+}\theta_{1-}} = \phi_{1\pm} \pm \left(\frac{r_{1+}r_{1-}}{\phi_{1+}-\phi_{1-}} + \frac{q_0 r_{1+}^2 r_{1-}^2}{(\phi_{1+}-\phi_{1-})^3}\right) + \frac{\theta_{1\mp}\theta'_{1\pm}}{1-\theta_{1+}\theta_{1-}}.$$

Here we remark that

$$\begin{aligned}\Re\left\{\frac{r_{1+}r_{1-}}{\phi_{1+}-\phi_{1-}}\right\} &= -\left(\frac{b'}{2c}-\frac{c'}{2c}\right)\left(\frac{b'}{2c}+\frac{c'}{2c}\right)\Re\left\{\frac{1}{-2\mathrm{i}\xi c-\frac{b'}{c}}\right\}\\ &= -\frac{1}{2\xi c}\left(\frac{b'}{2c}-\frac{c'}{2c}\right)\left(\frac{b'}{2c}+\frac{c'}{2c}\right)\Re\left\{\frac{\mathrm{i}}{1-\frac{\mathrm{i}b'}{2\xi c^2}}\right\}\\ &= -\frac{1}{2\xi c}\left(\frac{b'}{2c}-\frac{c'}{2c}\right)\left(\frac{b'}{2c}+\frac{c'}{2c}\right)\Re\left\{\mathrm{i}-\frac{b'}{2\xi c^2}-\mathrm{i}\left(\frac{b'}{2\xi c^2}\right)^2+\left(\frac{b'}{2\xi c^2}\right)^3+\cdots\right\}\\ &= -\frac{1}{2\xi c}\left(\frac{b'}{2c}-\frac{c'}{2c}\right)\left(\frac{b'}{2c}+\frac{c'}{2c}\right)\left(-\frac{b'}{2\xi c^2}+\left(\frac{b'}{2\xi c^2}\right)^3+\cdots\right)\in S^{(1)}\{-2,3\}.\end{aligned}$$

It follows that

$$\phi_{2\Re\pm}=\mp\frac{b'}{2c}+\frac{c'}{2c}+q_{1\pm},\quad q_{1\pm}\in S^{(0)}\{-2,3\}.$$

Therefore, we can apply Proposition 3.2 in Z_H since the condition (3.41) is satisfied. Thus we conclude the proof of Theorem 3.2. □

3.6 C^m property of the coefficient for wave type equations

In this section we discuss about some benefit from further smoothness of the coefficients.

Firstly we introduce a result that (3.20) is not valid.

Theorem 3.4 ([16])**.** *Let $\alpha>1$ and $a(t)=2+\cos\left((\log(1+t))^\alpha\right)$. Then (3.20) does not holds in general.*

The assumption in Theorem 3.1 for (3.20) is

$$\left|a^{(k)}(t)\right|\le C_k(1+t)^{-k}\quad(k=1,2).$$

On the other hand, Theorem 3.4 implies that for any $\alpha>1$ there exists $a(t)$ satisfying

$$\left|a^{(k)}(t)\right|\le C_k\left((1+t)^{-1}\left(\log(\mathrm{e}+t)\right)^{\alpha-1}\right)^{-k}\quad(k=1,2),$$

such that (3.20) does not hold. Thus the assumption (3.31) cannot be improved anymore even if $a(t)$ is smoother. Actually, if $a(t)\in C^m$ for $m\ge 3$, and satisfy some suitable assumptions, for instance

$$\left|a^{(k)}(t)\right|\le C_k(1+t)^{-k}\quad(k=1,\cdots,m),$$

then we can carry out further steps of diagonalization procedure. Consequently, we come to the following equation:

$$\partial_t V_m=(\Phi_m+R_m)V_m,\quad \Phi_m=\begin{pmatrix}\phi_{m+}&0\\0&\phi_{m-}\end{pmatrix},\quad R_m=\begin{pmatrix}0&r_{m+}\\r_{m-}&0\end{pmatrix},\tag{3.49}$$

where it will be that $r_{m\pm}\in S^{(0)}\{-m+1,m\}$. However, the property $r_{m\pm}\in S^{(0)}\{-m+1,m\}$ does not bring any benefit than the one from $r_2\in S^{(m-2)}\{-1,2\}$. On the other hand, it is also true that order of the anti-diagonal entries are smaller in consequence of further steps of diagonalization procedure in high frequency part. If one wants to derive a benefit from the better estimate in high frequency part conclude from C^m property of the coefficients and further steps of diagonalization procedure, the following condition, which is called the stabilization property is effective:

$$\int_0^t|a(\tau)-a_\infty|\,d\tau\le C(1+t)^\alpha.\tag{3.50}$$

Here $\alpha=1$ is a trivial case, hence (3.50) has a meaning only for $0\le\alpha<1$.

If we assume (3.50), then we have the following estimate in the low frequency pat:

$$\partial_t \mathcal{E}_\infty(t,\xi) = \left(a_\infty^2 - a(t)^2\right)\xi^2 \Re\{v\overline{v_t}\} \le \frac{\left|a_\infty^2 - a(t)^2\right||\xi|}{a_\infty}\mathcal{E}_\infty(t,\xi) \le C\left|a_\infty - a(t)\right|\mathcal{E}_\infty(t,\xi).$$

By Gronwall's lemma and (3.50) we have

$$\mathcal{E}_\infty(t,\xi) \le \mathcal{E}_\infty(0,\xi)\exp\left(C|\xi|\int_0^t |a_\infty - a(\tau)|\,d\tau\right) \le \mathcal{E}_\infty(0,\xi)\exp\left(C|\xi|(1+t)^\alpha\right).$$

Consequently, we have (3.21) in the following zone:

$$Z_{\Psi,\alpha} = \{(t,\xi)\in[0,\infty)\times\mathbb{R}\,;\,|\xi|(1+t)^\alpha \le N\}, \tag{3.51}$$

where N is a positive constant. Here we remark that $Z_\Psi \subset Z_{\Psi,\alpha}$, and $Z_{\Psi,\alpha_1}\subset Z_{\Psi,\alpha_2}$ for $\alpha_1 > \alpha_2$.

Let $m \ge 2$ and $a(t)\in C^m([0,\infty))$ satisfy the following conditions, which is a generalization of (3.31):

$$\left|a^{(k)}(t)\right| \le C_k(1+t)^{-\beta k} \quad (k=1,\cdots,m), \tag{3.52}$$

where $0\le\beta<1$, otherwise, (3.20) has already proved as Theorem 3.1. We define $Z_{H,\alpha}$ by

$$Z_{H,\alpha} = \{(t,\xi)\in[0,\infty)\times\mathbb{R}\,;\,|\xi|(1+t)^\alpha \ge N\}. \tag{3.53}$$

Then the symbol classes in $Z_{H,\alpha}$ should be introduced as follows:

$$f(t,\xi)\in S_\beta^{(m)}\{k,l\} \;\Leftrightarrow\; \left|\partial_t^j f(t,\xi)\right| \le C_j|\xi|^k(1+t)^{-l-j}. \tag{3.54}$$

Then the same properties of (i)-(iv) for the symbol class $S^{(m)}\{k,l\}$ are valid. The property corresponding (v) is given by

$$f(t,\xi)\in S_\beta^{(0)}\{-m+1,m\},\;\; \beta\ge\beta_m := \alpha + \frac{1-\alpha}{m} \;\Rightarrow\; \sup_{(t,\xi)\in Z_{H,\alpha}}\left\{\int_{t_\xi}^t |f(\tau,\xi)|d\tau\right\} < \infty, \tag{3.55}$$

where t_ξ is the curve which separates $Z_{\Psi,\alpha}$ from $Z_{H,\alpha}$ defined by $|\xi|(1+t_\xi)^\alpha = N$.

Remark 3.4. (i) If $0\le\alpha<1$, then $\beta_m = \alpha + \frac{1-\alpha}{m} < 1$.

(ii) β_m is monotone decreasing and $\lim_{m\to\infty}\beta_m = \alpha$.

(iii) If $m=1$ or $\alpha=1$, then $\beta\ge1$; this is the same assumption to the order of $a'(t)$ in Theorem 3.1. Actually, not C^2 but $C^{1,\varepsilon}$ ($\varepsilon>0$) regularity is essential to carry out our diagonalization procedure. Indeed, for any $\varepsilon>0$ we may prove the estimate (3.20) under the following assumptions:

$$|a'(t)| \le C_1(1+t)^{-1}, \quad \sup_{0<h<1}\frac{|a'(t+h)-a'(t)|}{h^\varepsilon} \le C_{1+\varepsilon}(1+t)^{-1-\varepsilon}$$

by using the same argument in [10].

If we introduce the stabilization property (3.50) and the C^m property (3.52) simultaneously, we have the following theorem, which drive a benefit of C^m regularity of the coefficient.

Theorem 3.5 ([8]). *Let $m\ge2$ be an integer, α and β be real numbers satisfying $0\le\alpha,\beta<1$. If $a(t)\in C^m([0,\infty))$ satisfies (3.50) and (3.52) for $\beta\ge\beta_m=\alpha+\frac{1-\alpha}{m}$, then (3.20) is established.*

A necessity of the condition $\beta\ge\beta_m$ for (3.20) is an interesting open problem. Incidentally, we have a result that (3.20) is not true in the limit case $\beta<\beta_\infty=\alpha$ as follows:

Theorem 3.6 ([8]). *Let α and β satisfy $0\le\beta<\alpha<1$. There exists $a(t)\in C^\infty([0,\infty))$ satisfying (3.50) and (3.52) such that (3.20) does not hold.*

Proof of Theorem 3.5. We have already proved the energy estimate (3.21) in $Z_{\Psi,\alpha}$; thus we consider the estimate in $Z_{H,\alpha}$. Let us consider the following first order system:

$$\partial_t V_1 = (\Phi_1 + R_1)V_1, \quad \Phi_1 = \begin{pmatrix} \phi_1 & 0 \\ 0 & \overline{\phi_1} \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & r_1 \\ \overline{r_1} & 0 \end{pmatrix}, \tag{3.56}$$

where $\phi_1 = -\mathrm{i}\xi a(t) + \frac{a'(t)}{2a(t)}$ and $r_1 = -\frac{a'}{2a}$. Then by (3.52) we have

$$\phi_1 \in S_\beta^{(m)}\{1,0\}, \quad r_1 \in S_\beta^{(m-1)}\{0,1\}, \quad \frac{1}{\phi_1} \in S_\beta^{(m)}\{-1,0\}. \tag{3.57}$$

Here we can define the diagonalizer Θ_1 of $\Phi_1 + R_1$ by

$$\Theta_1 = \begin{pmatrix} 1 & \overline{\theta_1} \\ \theta_1 & 1 \end{pmatrix}, \quad \theta_1 = \frac{\lambda_1 - \phi_1}{r_1}, \quad \lambda_1 = \phi_{1\Re} + \mathrm{i}\frac{\sqrt{|\phi_{1\Im}|^2 - 4|r_1|^2}}{2}. \tag{3.58}$$

Here we recall that the eigenvalues of $\Phi_1 + R_1$ are complex conjugate. Then we see that $\theta_1 \in S_\beta^{(m-1)}\{-1,1\}$, and thus Θ_1 is bounded from $\mathbb{C}^2$ to $\mathbb{C}^2$ and invertible uniformly in $Z_{H,\alpha}$. Denoting $V_2 = \Theta_1^{-1} V_1$ we have

$$\partial_t V_2 = (\Phi_2 + R_2)\, V_2, \quad \Phi_2 = \begin{pmatrix} \phi_2 & 0 \\ 0 & \overline{\phi_2} \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & r_2 \\ \overline{r_2} & 0 \end{pmatrix}, \tag{3.59}$$

where

$$r_2 = \frac{\overline{\theta_1}'}{1 - |\theta_1|^2}, \quad \phi_2 = \lambda_1 - \frac{\overline{\theta_1}\theta_1'}{1 - |\theta_1|^2}.$$

Generally we observe the following fact:

Proposition 3.3. *Let the matrix A_p have the following structure:*

$$A_p = \begin{pmatrix} \phi_p & r_p \\ \overline{r_p} & \overline{\phi_p} \end{pmatrix},$$

$\lambda_{p\pm}$ be the eigenvalues of A_p, and $(1, \theta_{p+})^T$ and $(\theta_{p-}, 1)^T$ be the corresponding eigenvectors. If $1/\phi_p \in S_\beta^{(j)}\{-1,0\}$ and $r_p \in S_\beta^{(j)}\{-l+1,l\}$ for $j \geq 1$ and $l \geq 1$, then we have the following properties for large N:

(i) $\lambda_{p+} = \overline{\lambda_{p-}} =: \lambda_p$.

(ii) The diagonalizer of Θ_p of A_p is given by

$$\Theta_p = \begin{pmatrix} 1 & \overline{\theta_p} \\ \theta_p & 1 \end{pmatrix}, \quad \theta_p = \frac{\lambda_p - \phi_p}{r_p} \in S_\beta^{(j)}\{-l,l\},$$

hence Θ_p is bounded and invertible in $Z_{H,\alpha}$.

(iii) $A_{p+1} := \Theta_p^{-1} A_p \Theta_p - \Theta_p^{-1}(\partial_t \Theta_p)$ *has the following representation:*

$$A_{p+1} = \begin{pmatrix} \phi_{p+1} & r_{p+1} \\ \overline{r_{p+1}} & \overline{\phi_{p+1}} \end{pmatrix}, \quad r_{p+1} = \frac{\overline{\theta_p}'}{1 - |\theta_p|^2}, \quad \phi_{p+1} = \lambda_p - \frac{\overline{\theta_p}\theta_p'}{1 - |\theta_p|^2}. \tag{3.60}$$

More precisely we have

$$\phi_{p+1} = \lambda_p - \frac{1}{2}\left(\log\left(1 - |\theta_p|^2\right)\right)' + \mathrm{i}\frac{\Im\{\overline{\theta_p}\theta_p'\}}{1 - |\theta_p|^2}. \tag{3.61}$$

It follows that

$$\Re\left\{\phi_{p+1} - \lambda_p\right\} = -\frac{1}{2}\left(\log\left(1 - |\theta_p|^2\right)\right)'. \tag{3.62}$$

(iv) $r_{p+1} \in S_\beta^{(j-1)}\{-l, l+1\}$.

This proposition can be proved by direct computations.

By applying Proposition 3.3 to (3.59), we come to the following system:

$$\partial_t V_m = (\Phi_m + R_m) V_m, \quad V_m = \Theta_{m-1}^{-1} \cdots \Theta_1^{-1} V_1, \quad \Phi_m = \begin{pmatrix} \phi_m & 0 \\ 0 & \overline{\phi_m} \end{pmatrix}, \quad R_m = \begin{pmatrix} 0 & r_m \\ \overline{r_m} & 0 \end{pmatrix}, \tag{3.63}$$

where $r_m \in S_\beta^{(0)}\{-m+1, m\}$ and

$$\Re\{\phi_m\} = \Re\{\lambda_1\} - \frac{1}{2} \sum_{k=1}^{m-1} \left(\log\left(1 - |\theta_k|^2\right)\right)' = \frac{1}{2} \partial_t \log \left(\frac{a}{\prod_{k=1}^{m-1} (1 - |\theta_k|^2)} \right), \tag{3.64}$$

it follows that

$$\int_s^t \Re\{\phi_m(\tau, \xi)\} d\tau = \frac{1}{2} \partial_t \log \left(\frac{a(t) \prod_{k=1}^{m-1} \left(1 - |\theta_k(t,\xi)|^2\right)}{a(s) \prod_{k=1}^{m-1} (1 - |\theta_k(s,\xi)|^2)} \right).$$

Therefore, $\int_s^t \Re\{\phi_m(\tau, \xi)\} d\tau$ is uniformly bounded in $Z_{H,\alpha}$. Consequently, we can apply Proposition 3.2 in $Z_{H,\alpha}$, and thus the proof of Theorem 3.5 is concluded. □

Remark 3.5. Theorem 3.2 is generalized in [2] taking account of the C^m and the stabilization properties of the coefficients.

3.7 Gevrey property of the coefficient for wave type equations

It may be a natural observation that further smoothness property than C^m, for instance C^∞ or Gevrey regularity of the coefficient brings some benefit.

Let introduce more precise the condition of (3.52) for $a(t) \in C^\infty([0, \infty))$ as follows:

$$\left| a^{(k)}(t) \right| \le C k!^\nu \left((1+t)^\alpha \left(\log(\mathrm{e}+t) \right)^\delta \right)^{-k} \quad (k = 1, 2, \cdots). \tag{3.65}$$

If (3.52) holds for $\beta > \alpha$, then Theorem 3.5 is applicable. Hence it should be that $\beta = \alpha$, and logarithmic order decay is introduced in (3.65). Then we the following theorem:

Theorem 3.7 (Hirosawa '10)**.** *Let $\alpha \in [0, 1)$, $\delta \ge 0$ and $\nu \ge 1$. If $a(t) \in C^\infty([0, \infty))$ satisfies (3.50) and (3.65) for $\nu \le \delta$, then (3.20) is valid.*

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