On the global solvability for semilinear wave equations with smooth propagation speeds

F. Hirosawa (Yamaguchi University, Japan)

joint work with

T. Inooka (Yamaguchi University)
and
T. D. Pham (Hanoi National University of Education)

1. Introduction

Cauchy problem of non-linear wave equation with time dependent propagation speed:

$$(\partial_t^2 - a(t)^2\Delta)u = F(t, \partial_t u, \nabla u), \ (t, x) \in \boldsymbol{R}_+ \times \boldsymbol{R}^n$$

$$(u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \ x \in \boldsymbol{R}^n$$

u: real valued, $u_0, u_1 \in C_0^\infty(\boldsymbol{R}^n)$, $n \geq 2$,
$a(t) \geq a_0 > 0$, $F(t, 0,0) = 0$.

Problem

Existence of a global solution with small data (=GS) under the influences of the variable coefficient

2. Background

I. Constant coefficient

$$(\partial_t^2 - \Delta)u = F(\partial_t u, \nabla u),\ F(0,0) = 0$$

(GS) is valid under suitable assumptions to the nonlinear terms $F(p,q)$, the initial data (u_0, u_1) and the space dimension n.

Linear wave equation

$$(\partial_t^2 - \Delta)u = 0$$

The properties: energy conservation, L^p-L^q estimates, etc., are established.

II. Linear wave equation with variable coefficient

$$(\partial_t^2 - a(t)^2\Delta)u = 0$$

Generally, we cannot expect the properties: the energy conservation, L^2 well–poseness, L^p-L^q estimates, etc.

Energy estimate

$$E(t) = \frac{1}{2}\left(a(t)^2\|\nabla u(t,\cdot)\|_{L^2}^2 + \|\partial_t u(t,\cdot)\|_{L^2}^2\right)$$

$$\Rightarrow E'(t) = a'(t)a(t)\|\nabla u(t,\cdot)\|_{L^2}^2 \leq \frac{2|a'(t)|}{a(t)}E(t)$$

$$\Rightarrow E(t) \leq \exp\left(2\int_0^t \frac{|a'(s)|}{a(s)}ds\right)E(0)$$

Boundedness of the energy is not trivial if $a(t)$ is oscillating infinitely many times.

$\underline{0 < a_0 \leq a(t) \leq a_1}$

- $a'(t) \in L^1(\boldsymbol{R}_+) \Rightarrow E(t) \simeq E(0)$

- $a(t)$: periodic

  $\Rightarrow \exists (u_0, u_1)$, s.t. $\limsup_{t \to \infty} E(t) = \infty$ [Yagdjian '05]

- $a(t) \in C^2(\boldsymbol{R})$, $|a'(t)|^2 + |a''(t)| \leq C(1+t)^{-2}$

  $\Rightarrow E(t) \simeq E(0)$ [Reissig & Smith '05]

- $a(t) \notin C^1(\boldsymbol{R}_+)$, $a(t) \in C^\alpha(\boldsymbol{R}_+)$, $\alpha \in (0,1)$

  $\Rightarrow$ Gevrey $1/(1-\alpha)$ well-posed

  [Colombini, DeGiorgi & Spagnolo '79]

$\underline{a(t) \geq 0}$

- $a(t) \in C^{m,\alpha}(\boldsymbol{R}_+)$, $\alpha \in (0,1)$, $m \in \boldsymbol{Z}$, $m \geq 0$

 $\Rightarrow$ Gevrey $1 + (m + \alpha)/2$ well−posed

 [Colombini, Jannelli & Spagnolo '84]

Remarks.

- $f(x)$: Gevrey class of order $s \Leftrightarrow \left|\hat{f}(\xi)\right| \leq \exp\left(\rho\langle\xi\rangle^{1/s}\right)$
- Linear wave equations with time dependent coefficients are studied as linearized model of Kirchhoff equation:

$$\partial_t^2 u + \left(1 + \int |\nabla u(t,x)|^2 dx\right)\Delta u = 0$$

[Arosio & Spagnolo '84], [Manfrin '05], [H. '02, '06]

III. Monotone increasing coefficients

$$(\partial_t^2 - \lambda(t)^2\Delta)u = 0,\ \lambda(0) > 0,\ \lambda'(t) \geq 0$$

Energy estimate

$$E'(t) = \lambda'(t)\lambda(t)\|\nabla u(t,\cdot)\|_{L^2}^2 \leq \frac{2\lambda'(t)}{\lambda(t)}E(t)$$

$$\Rightarrow\ E(t) \leq \frac{\lambda(t)^2}{\lambda(0)^2}E(0)$$

- L^p-L^q estimates [Reissig & Yagdjian ’00], etc.

IV. Mixed case

$$a(t) = \lambda(t) b(t)$$

$$\lambda(0) > 0,\ \lambda'(t) \geq 0,\ 0 < b_0 \leq b(t) \leq b_1$$

- $b(t) \in C^2(\boldsymbol{R}_+)$: periodic, $\lambda(t) = \exp(t^\alpha)$, $\alpha > 1/2$

  $\Rightarrow L^p$-L^q estimate [Reissig & Yagdjian '00]

- $b(t) \in C^2(\boldsymbol{R}_+)$: periodic, $\lambda(t) = \exp(t^\alpha)$, $\alpha = 0$

  $\Rightarrow$ no L^p-L^q estimate

Remark. Faster increasing $\lambda(t)$ has a good influence for the estimate.

Properties of the coefficient

$$a(t) = \lambda(t)b(t)$$
$$\lambda(0) > 0,\ \lambda'(t) \geq 0,\ 0 < b_0 \leq b(t) \leq b_1$$

- smoothness: $a(t) \in C^m(\boldsymbol{R}_+),\ m \geq 1$
- order of the derivatives:

$$\left|b^{(k)}(t)\right| \leq C_k\, \rho(t)^k,\ (k = 1, \cdots, m)$$

- increasing order: $\lambda(t)$
- stabilization (difference from a monotone function)

$$\int_0^t |a(s) - \lambda(s)| ds \leq \theta(t) = o\left(\int_0^t \lambda(s) ds\right) = o(\Lambda(t))$$

In this talk we restrict ourselves

$$\rho(t) = \frac{\lambda(t)}{\Lambda(t)} (\log \Lambda(t))^{\beta}, \quad \theta(t) = \Lambda(t)(\log \Lambda(t))^{-\gamma}$$

for $\beta, \gamma \geq 0$ and $t \gg 1$, that is,

$$\left|b^{(k)}(t)\right| \leq C_k \left(\frac{\lambda(t)}{\Lambda(t)} (\log \Lambda(t))^{\beta}\right)^k, (k = 1, \cdots, m)$$

$$\int_0^t |a(s) - \lambda(s)| ds \leq \Lambda(t)(\log \Lambda(t))^{-\gamma}$$

Remark. $\gamma = 0$ is a trivial case.

Example 1. $\lambda(t) = \exp(t^{\alpha}),\ b(t) \in C^m$, periodic.

$$\begin{cases} \Lambda(t) \simeq t^{1-\alpha}\lambda(t) \\ \left|b^{(k)}(t)\right| \le C_k \left(\dfrac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^{\beta}\right)^k \simeq t^{k(\alpha-1+\alpha\beta)} \end{cases}$$

$\Rightarrow \alpha - 1 + \alpha\beta \ge 0,\ \gamma = 0$

Example 2. $\lambda(t) = (1+t)^p$, $b(t) \in C^m$, periodic.

$$\begin{cases} \Lambda(t) \simeq t\,\lambda(t) \\ \left|b^{(k)}(t)\right| \le C_k \left(\dfrac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^{\beta}\right)^k \simeq \left(t^{-1}(\log t)^{\beta}\right)^k \end{cases}$$

such a constant β does not exist.

Example 3.

$$\lambda(t) = (1+t)^p,\ b(t) = 2 + \cos((\log(1+t))^\delta)$$

$$\begin{cases} \Lambda(t) \simeq t\,\lambda(t),\ \left|b^{(k)}(t)\right| \le C_k\left(t^{-1}(\log t)^{\delta-1}\right)^k \\ \dfrac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^\beta \simeq t^{-1}(\log t)^\beta \end{cases}$$

$\Rightarrow \delta - 1 \le \beta,\ \gamma = 0$

Example 4. $\lambda(t) = \exp(t^{\alpha})$,

$$b(t) = \begin{cases} p(t) & t \in I_j = [j^{1/\alpha}, j^{1/\alpha} + 1] \\ 1 & t \in [0,\infty) \setminus \cup_{j=1}^{\infty} I_j \end{cases}$$

$p(t) \in C^m$, 1−periodic, $p(t) > 0$, $p^{(k)}(t) = 0$ near $t = 0$ for $k = 1, \cdots, m$.

$$\begin{cases} \Lambda(t) \simeq t^{1-\alpha}\lambda(t) \\ \left|b^{(k)}(t)\right| \leq C_k \left(\frac{\lambda(t)}{\Lambda(t)} (\log \Lambda(t))^{\beta}\right)^k \simeq t^{k(\alpha-1+\alpha\beta)} \end{cases}$$

$\Rightarrow \alpha - 1 + \alpha\beta \geq 0,\ \gamma = 1/\alpha - 1$

V. Nonlinear equations

We restrict ourselves to the following special nonlinearity:

$$F(p, q) = |p|^2 - |q|^2$$

$$(\partial_t^2 - \Delta)u = |\partial_t u|^2 - |\nabla u|^2, \ (u_0(x), u_1(x))$$

$$\Updownarrow \quad v = 1 - \exp(-u) \quad \text{(Nirenberg's transformation)}$$

$$(\partial_t^2 - \Delta)v = 0, \ (v_0(x), v_1(x)) = (1 - e^{-u_0}, u_1 e^{-u_0})$$

$$0 < v(t, x) < 1, \ t \in [0, T),$$

$$\lim_{t \to T-0} v(t, x) = 1 \ \Rightarrow \ \lim_{t \to T-0} u(t, x) = \infty$$

$$(\partial_t^2 - a(t)^2\Delta)u = |\partial_t u|^2 - a(t)^2|\nabla u|^2$$

$\Updownarrow$ $v = 1 - \exp(-u)$ (Nirenberg's transformation)

$$(\partial_t^2 - a(t)^2\Delta)v = 0$$

$a(t) \equiv a_0 \Rightarrow |v(t,x)| < 1$ if $|v_0(x)| + |v_1(x)| \ll 1$

$a(t) \not\equiv const. \Rightarrow |v(t,x)| < 1$ does not follow from $|v_0(x)| + |v_1(x)| \ll 1$

$$(\partial_t^2 - a(t)^2\Delta)u = |\partial_t u|^2 - a(t)^2|\nabla u|^2$$

$$\left|b^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)}(\log \Lambda(t))^\beta\right)^k, (k = 1, \cdots, m)$$

Theorem. ([Reissig & Yagdjian '00]).

$m \ge 2$, $\beta < 1$ $\Rightarrow$ (GS).

Corollary.

$\lambda(t) = \exp(t^\alpha)$, $b(t) \in C^2$, *periodic*, $\alpha > 1/2$
$\Rightarrow$ (GS).

Remark. (GS) for $\alpha \in (0, 1/2)$ is an open problem.

3. Main Theorem

$$(\partial_t^2 - a(t)^2\Delta)u = |\partial_t u|^2 - a(t)^2|\nabla u|^2$$

$$\left|b^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)}(\log \Lambda(t))^{\beta}\right)^k, \ (k = 1, \cdots, m)$$

$$\int_0^t |a(s) - \lambda(s)| ds \le \Lambda(t)(\log \Lambda(t))^{-\gamma}$$

Motivation We want to derive benefits of the following properties of the coefficients for (GS):

- further smoothness: $m \ge 3$;
- stabilization $\gamma > 0$.

$$\left|b^{(k)}(t)\right| \leq C_k \left(\frac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^{\beta}\right)^k, (k = 1, \cdots, m)$$

$$\int_0^t |a(s) - \lambda(s)|ds \leq \Lambda(t)(\log\Lambda(t))^{-\gamma}$$

Theorem. ([H., Inooka & Pham]).

$m \geq 2$, $\beta < 1 + \gamma(1 - 1/m)$ ⇒ (GS).

Corollary.

$\lambda(t) = \exp(t^{\alpha})$, $b(t)$: Example 4, $\alpha > 1/(m+1)$ ⇒ (GS).

Comparison with the previous results

$$\left|b^{(k)}(t)\right| \le C_k \left(\frac{\lambda(t)}{\Lambda(t)}(\log\Lambda(t))^{\beta}\right)^k, \ (k = 1, \cdots, m)$$

$$\int_0^t |a(s) - \lambda(s)| ds \le \Lambda(t)(\log\Lambda(t))^{-\gamma}$$

	[HIP]	[RY]
General model ($\beta <$)	$1 + \gamma\left(1 - \frac{1}{m}\right)$	1
Ex. 4 ($\alpha >$)	$\frac{1}{m+1}$	$\frac{1}{2}$

4. Sketch of the proof

$$(\partial_t^2 - a(t)^2\Delta)v = 0, \quad v(0,x) = v_0, v_t(0,x) = v_1$$

Proposition 1. *(GS) is valid if the following estimate holds for $s > n$:*

$$\sup_{t,\xi}\{|v(t,x)|\} \leq C(\|v_0\|_{W^{s,1}} + \|v_1\|_{W^{s-1,1}}) \qquad (*)$$

$w(t,\xi) = \hat{v}(t,\xi)$

$$(\partial_t^2 + a(t)^2|\xi|^2)w = 0, \quad w(0,\xi) = w_0, w_t(0,\xi) = w_1$$

(∗) is valid if the following estimate holds:

$$|w(t,\xi)| \leq C|\xi|^{-\kappa}(|w_0(\xi)| + |\xi|^{-1}|w_1(\xi)|) \qquad (**)$$

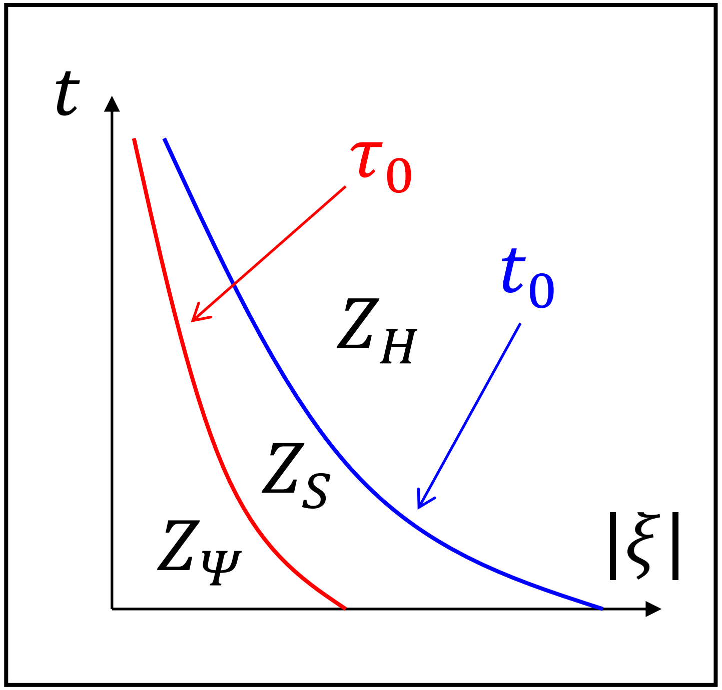
$$\tau_0 \lambda(\tau_0)|\xi| = N(\gg 1)$$

$$\Lambda(t_0)|\xi| = N(\log \Lambda(t_0))^{\gamma+1}$$

$$Z_\Psi = \{(t,\xi)\,;\ 0 \le t \le \tau_0\}$$

$$Z_S = \{(t,\xi)\,;\ \tau_0 \le t \le t_0\}$$

$$Z_H = \{(t,\xi)\,;\ t_0 \le t\}$$



Estimate in Z_H

Reduction to 1st order system

$$(\partial_t^2 + a(t)^2|\xi|^2)w = 0$$

$$\Downarrow$$

$$\partial_t W_1 = A_1 W_1$$

$$W_1 = \begin{pmatrix} w_t + ia|\xi|w \\ w_t - ia|\xi|w \end{pmatrix},\ \ A_1 = \begin{pmatrix} \phi_1 & \overline{b_1} \\ b_1 & \overline{\phi_1} \end{pmatrix}$$

$$\phi_1 = \frac{a'}{2a} + ia|\xi|,\ b_1 = \overline{b_1} = \frac{a'}{2a}$$

$$\lambda(t)|\xi||w(t,\xi)| + |w_t(t,\xi)| \simeq |W_1(t,\xi)|$$

Diagonalization

$$M_1^{-1}A_1M_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix},\ M_1 = \begin{pmatrix} 1 & \overline{\delta_1} \\ \delta_1 & 1 \end{pmatrix}$$

$$\partial_t W_1 = A_1 W_1$$

$$\Downarrow$$

$$\partial_t W_2 = A_2 W_2,\ \ W_2 = M_1^{-1} W_1$$

$$A_2 = \begin{pmatrix} \phi_2 & \overline{b_2} \\ b_2 & \overline{\phi_2} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \overline{\lambda_1} \end{pmatrix} - M_1^{-1}(\partial_t M_1)$$

Symbol classes in Z_H

$$f(t,\xi) \in S^{(p)}\{q,r\}$$

$$\Leftrightarrow \left|\partial_t^k f(t,\xi)\right| \leq C_k(\lambda|\xi|)^q \left(\frac{\lambda}{\Lambda}(\log\Lambda)^\beta\right)^{r+k} \quad (k = 1,\cdots,p)$$

Lemma. $f \in S^{(p)}\{q,r\}$, $g \in S^{(p)}\{q',r'\} \Rightarrow$

(i) $\partial_t f \in S^{(p-1)}\{q, r+1\}$;

(ii) $fg \in S^{(p)}\{q+q', r+r'\}$;

(iii) $S^{(p)}\{q,r\} \subset S^{(p)}\{q+1, r-1\}$;

(iv) $b_1 \in S^{(m-1)}\{0,1\}$, $1/\phi_{1\mathfrak{J}} \in S^{(m)}\{-1,0\}$, $\phi_{1\mathfrak{J}} > 0$.

Proposition 2.

$$A_k = \begin{pmatrix} \phi_k & \overline{b_k} \\ b_k & \overline{\phi_k} \end{pmatrix},\ M_k = \begin{pmatrix} 1 & \overline{\delta_k} \\ \delta_k & 1 \end{pmatrix},\ \delta_k = \frac{\lambda_k - \phi_k}{\overline{b_k}}$$

$$b_k \in S^{(m-k)}\{-k+1, k\},\ 1/\phi_{k\mathfrak{I}} \in S^{(m-k)}\{-1,0\},\ \phi_{k\mathfrak{I}} > 0$$

$$\Rightarrow A_{k+1} = M_k^{-1}(A_k - \partial_t)M_k$$

$$= \begin{pmatrix} \lambda_k & 0 \\ 0 & \overline{\lambda_k} \end{pmatrix} + M_k^{-1}(\partial_t M_1) = \begin{pmatrix} \phi_{k+1} & \overline{b_{k+1}} \\ b_{k+1} & \overline{\phi_{k+1}} \end{pmatrix}$$

$$b_{k+1} \in S^{(m-k-1)}\{-k, k+1\},$$

$$1/\phi_{(k+1)\mathfrak{I}} \in S^{(m-k-1)}\{-1,0\},\ \phi_{(k+1)\mathfrak{I}} > 0,$$

$$\phi_{(k+1)\mathfrak{R}} = \phi_{k\mathfrak{R}} - \frac{\partial_t \log\left(1 - |\delta_k|^2\right)}{2}$$

After $m-1$ steps diagonalization

$$\partial_t W_m = A_m W_m$$

$$W_m = M_{m-1}^{-1} \cdots M_1^{-1} W_1, \quad A_m = \begin{pmatrix} \phi_m & \overline{b_m} \\ b_m & \overline{\phi_m} \end{pmatrix}$$

$$b_m \in S^{(0)}\{-m+1, m\},\ \phi_{m\Re} = \frac{1}{2}\partial_t \log\left(\frac{a}{\prod_{k=1}^{m-1}(1-|\delta_k|^2)}\right)$$

$$\partial_t |W_m|^2 = 2Re(A_m W_m, W_m) \leq 2(\phi_{m\Re} + |b_m|)|W_m|^2$$

$$\Rightarrow |W_m(t,\xi)|^2 \leq \frac{a(t)}{a(t_0)} \prod_{k=1}^{m-1} \frac{1-|\delta_k(t_0)|^2}{1-|\delta_k(t)|^2} \times \exp\left(\int_{t_0}^{t} 2|b_m(s,\xi)|ds\right) |W_m(t_0,\xi)|^2$$

$$\int_{t_0}^{t} |b_m(s,\xi)| ds \le C|\xi|^{-m+1} \int_{t_0}^{t} \frac{\lambda}{\Lambda^m} (\log \Lambda)^{m\beta} ds$$

$$\le C(\log \Lambda(t_0))^{-(\gamma+1)(m-1)+m\beta}$$

$$= o(\log \Lambda(t_0))$$

$$\Leftarrow -(\gamma+1)(m-1) + m\beta < 1 \quad \Leftrightarrow \beta < 1 + \gamma\left(1 - \frac{1}{m}\right)$$

$$\lambda(t)|\xi||w(t,\xi)| + |w_t(t,\xi)| \simeq |W_1(t,\xi)| \simeq |W_m(t,\xi)|$$

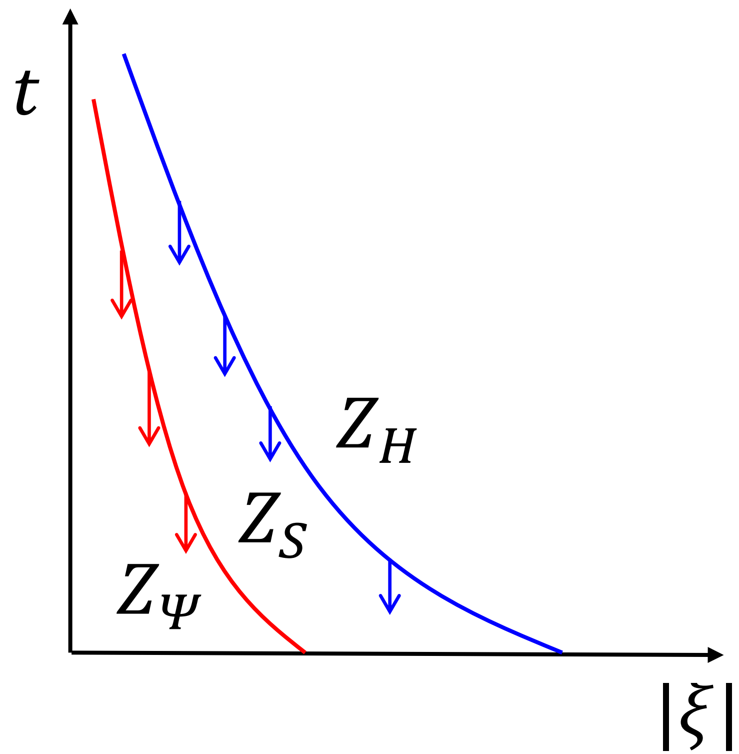
$$\le C\sqrt{\frac{\lambda(t)}{\lambda(t_0)}} \exp\left(\int_{t_0}^{t} |b_m(s,\xi)| ds\right) |W_m(t_0,\xi)|$$

$$\le C\sqrt{\frac{\lambda(t)}{\lambda(t_0)}} |W_m(t_0,\xi)| \simeq \sqrt{\frac{\lambda(t)}{\lambda(t_0)}} |W_1(t_0,\xi)|$$

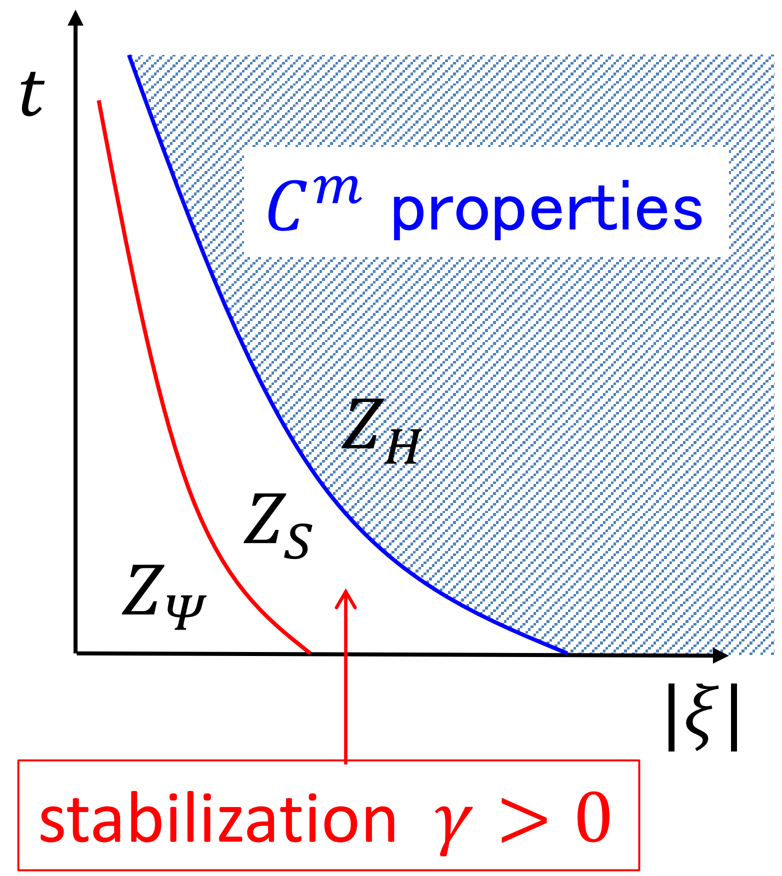
$$\tau_0 \lambda(\tau_0)|\xi| = N$$

$$\Lambda(t_0)|\xi| = N(\log \Lambda(t_0))^{\gamma+1}$$

<u>λ: larger</u>



<u>Influence of the properties</u>



5. Concluding remarks

Theorem ([H., Inooka & Pham]).

$m \geq 2$, $\beta < 1 + \gamma(1 - 1/m)$ $\Rightarrow$ (GS).

- It will be prove that (GS) is not true if $\beta > 1 + \gamma$ (cf. [H. '07]).
- Necessity of the smoothness is open problem.
- Application to Kirchhoff equation.
- Application to weakly hyperbolic problems.