On second order weakly hyperbolic equations and the ultradifferentiable classes

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joint work with

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Cauchy problem of second order weakly hyperbolic equation:

$$\begin{cases} \left(\partial_t^2 - a(t)^2\Delta\right) u(t,x) = 0, \quad (t,x) \in (0,T] \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), \quad x \in \mathbb{R}^n, \end{cases} \qquad (P)$$

$a \in C^\infty([0,T))$, $a(t) > 0$, $t \in [0,T)$, $a(T) = 0$.

- $\lambda \in C^1([0,T])$, $\lambda'(t) \le 0$, $\lambda(t) > 0$, $t \in [0,T)$, $\lambda(T) = 0$.
- $a(t) \simeq \lambda(t)$ $(\Leftrightarrow C_0\lambda(t) \le a(t) \le C_1\lambda(t))$.
- $\Lambda(t) := \int_t^T \lambda(s)\,ds$, $\Theta(t) := \int_t^T |a(s) - \lambda(s)|\,ds$.
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Our goal is to derive a relation between the order of μ for the estimate

$$\mathcal{E}(t,\xi) \le \exp(C\mu(\langle\xi\rangle))\mathcal{E}(0,\xi), \qquad (E)$$

$$\mathcal{E}(t,\xi) := \lambda(t)^2|\xi|^2|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2$$

and the properties of $a(t)$ represented by $\lambda(t)$, $\Theta(t)$ and $\{M_k\}$.

Known results (non-degenerate case)

$a(t) \in C^m([0,T))$, $\lambda(t) = \lambda_0 > 0$, $\Theta(t) \simeq T - t$:

- $m = 1$, $\rho(t) = (T-t)^{-1}$
  $\Rightarrow \mu(r) = \log r$ [Colombini-DelSanto-Kinoshita '02].
- $m = 2$, $\rho(t) = (T-t)^{-1}$
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If m and α larger (the coefficient is smoother and stabilized faster to a constant λ_0), then larger β (faster oscillation) is admited for the estimate (E) with $\mu(r) = 1$.

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Theorem (Summary of the previous results)

Let $a(t) \in C^m([0,T))$, $m \geq 2$. *If there exists* $\lambda(t) \in C^1([0,T])$ *satisfying* $\lambda(t) > 0$ *and* $\lambda'(t) \leq 0$ *such that* $a(t) \simeq \lambda(t)$ *and*

$$\left|a^{(k)}(t)\right| \leq \lambda(t) M_k \left(\lambda(t) \left(\frac{\Lambda(t)}{\Theta(t)} \right)^{-\frac{1}{m}} \left(\frac{1}{\Theta(t)} \right)^{\gamma} \right)^k$$

for $k = 1, \cdots, m$, *then the following estimate is valid:*

$$\mathcal{E}(t,\xi) \leq \exp(C\mu(\langle\xi\rangle))\mathcal{E}(0,\xi) \quad \textit{with} \quad \mu(r) = r^{\frac{\gamma-1}{\gamma}}.$$

Example

$a(t) = \lambda(t)\omega(t)$, $\lambda(t) = (T-t)^p$, $\Lambda(t) \simeq (T-t)^{p+1}$, $p \geq 0$,
$\nu > \kappa > 1$, $0 < \varepsilon < 1$, $\varepsilon^{-\nu+\kappa} \in \mathbb{N}$,
$\psi \in C^m(\mathbb{R})$: 1-periodic, $\phi(r) \equiv$ naer $r = 1$, $\tau_j := \varepsilon^j$,

$$\omega(t) := \begin{cases} \psi\left(\tau_j^{-\nu}(T-t-\tau_j)\right)^{\frac{1}{2}}, & t \in I_j := [T-\tau_j-\tau_j^{\kappa}, T-\tau_j+\tau_j^{\kappa}], \\ 1, & t \in [0,\infty) \setminus \bigcup_{j=1}^{\infty} I_j. \end{cases}$$

$$\int_t^T |a(\tau) - \lambda(\tau)|\, d\tau = \Theta(t) \simeq (T-t)^{p+\kappa},$$

$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \lesssim (T-t)^{-k\nu} \lesssim \left(\lambda(t)\left(\frac{\Lambda(t)}{\Theta(t)}\right)^{-\frac{1}{m}}\left(\frac{1}{\Theta(t)}\right)^{\gamma}\right)^{k}$$

for $k = 1, \cdots, m$ and $\gamma \leq \frac{p+\nu-\frac{1-\kappa}{m}}{p+k}$, it follows that

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$$\frac{\left|a^{(k)}(t)\right|}{\lambda(t)} \lesssim (T-t)^{-k\nu} \lesssim \left(\lambda(t)\left(\frac{\Lambda(t)}{\Theta(t)}\right)^{-\frac{1}{m}}\left(\frac{1}{\Theta(t)}\right)^{\gamma}\right)^k$$

for $k = 1, \cdots, m$ and $\gamma \leq \frac{p+\nu-\frac{1-\kappa}{m}}{p+k}$, it follows that

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with $\gamma = \frac{p+\nu-\frac{1-\kappa}{m}}{p+k}$ for $k = 1, \cdots, m$ where

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Remark

- s larger as p, κ, m and $1/\nu$ larger.
- The estimate is optimal for $p = 0$ and $m \to \infty$.

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$$\mathcal{M}(\tau) = \tau^m \quad \text{and} \quad \eta(r) = r^{\frac{s}{s-1}}$$

we have

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- The order of $\mathcal{M}$ should be faster than any polynomial order.
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Theorem ([H.-Ishida '13])

Let $\{M_k\}$ *be a logarithmic convex sequence:* $\frac{M_k}{kM_{k-1}} \le \frac{M_{k+1}}{(k+1)M_k}$, *and* $\mathcal{M}$ *be the associated function of* $\{M_k\}$, *which defined by*

$$\mathcal{M}(r) := \sup_{k\ge 1}\left\{\frac{r^k}{M_k}\right\} \quad \text{for} \quad r \ge 0.$$

If $a \in C^\infty$ *satisfy* $a(t) \simeq \lambda(t)$ *and*

$$\left|a^{(k)}(t)\right| \le \lambda(t) M_k \left(\frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)}\eta\left(\frac{1}{\Theta(t)}\right)\right)^k \quad (k \in \mathbb{N}),$$

then we have

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Examples

$\eta(\tau) = \tau^{\frac{s}{s-1}}$, $\mu(r) = r^{\frac{1}{s}}$; $\mathcal{E}(t,\xi) \le \mathcal{E}(0,\xi)\exp(C\langle\xi\rangle^{\frac{1}{s}})$; (Gevrey well-posedness).

- $M_k = k!^s$ $(s \ge 1)$:

$$\exp\left(C_0 r^{\frac{1}{s}}\right) \le \mathcal{M}(r) \le \exp\left(C_1 r^{\frac{1}{s}}\right),$$
$$C_0' (\log\tau)^s \le \mathcal{M}^{-1}(\tau) \le C_1' (\log\tau)^s.$$

- $\prod_{j=1}^{k} \exp(j^\kappa)$ $(\kappa > 0)$:

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$$\rho(t) = \frac{\lambda(t)}{\mathcal{M}^{-1}\left(\frac{\Lambda(t)}{\Theta(t)}\right)} \left(\frac{1}{\Theta(t)}\right)^{\frac{s}{s-1}},$$

$$\left|a^{(k)}(t)\right| \le \lambda(t) M_k \rho(t)^k \quad (k = 1, \cdots, m),$$

$$\Rightarrow \ \mathcal{E}(t,\xi) \le \mathcal{E}(0,\xi) \exp\left(C\langle\xi\rangle^{\frac{1}{s}}\right).$$

M_k	$\mathcal{M}(r)$	$\mathcal{M}^{-1}(\tau)$
$m < \infty$	r^m	$\tau^{\frac{1}{m}}$
$\prod_{j=1}^{k} \exp(j^\kappa)$	$\exp\left(C(\log r)^{\frac{\kappa+1}{\kappa}}\right)$	$\exp\left(C'(\log \tau)^{\frac{\kappa}{\kappa+1}}\right)$
$k!^s$	$\exp\left(Cr^{\frac{1}{s}}\right)$	$(\log \tau)^s$
$k!$	$\exp(Cr)$	$\log \tau$

Sketch of the proof

- Partial Fourier transformation; $\mathbb{R}^{n+1}_{t,x} \to \mathbb{R}^{n+1}_{t,\xi}$.
- Division of the phase space into infinitely many zones.
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Open problems

- Optimality: for instance, for $a(t) \in C^m([0,T))$, $\lambda(t) = 1$, $\Theta(t) = (T-t)^\alpha$, $|a^{(k)}(t)| \le M_k(T-t)^{-\beta k}$ $(k = 1, \cdots, m)$:
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