A class of non-analytic functions for the global solvability of Kirchhoff equation

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Kirchhoff equation

Consider the global solvability to the Cauchy problem of Kirchhoff equation:

$$\begin{cases} \left(\partial_t^2 - \Phi(t;u)\Delta\right) u(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (u(0,x), u_t(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^n, \end{cases} \tag{K}$$

where

$$\Phi(t;u) = 1 + \int_{\mathbb{R}^n} |\nabla_x u(t,x)|^2 \, dx.$$

Known results:

- Local solvability in Sobolev class [Bernstein '40].
- Global solvability in realanalytic (quasianalytic) class [Bernstein '40] ([Nishihara '84]).
- Global solvability with small data [Greenberg-Hu '80].

Basic observation

The solution to (K) has the following properties:

Energy conservation:

$$E(t) := \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\int_0^{\|\nabla u(t,\cdot)\|^2} (1+\eta)\,d\eta \equiv E(0).$$

L^2 boundedness:

$$\|u_t(t,\cdot)\|^2 + \|\nabla u(t,\cdot)\|^2 \leq 2E(0).$$

Local solvability in H^m

Define the higher order hyperbolic energy:

$$E_m(t) := \frac{1}{2}\|u_t(t,\cdot)\|_{H^m}^2 + \frac{1}{2}\Phi(t;u)\|\nabla u(t,\cdot)\|_{H^m}^2.$$

Then we have the following estimates, which imply the existence of a time local solution with $m \geq 1$:

$$\begin{aligned}\frac{d}{dt}E_m(t) &= \frac{1}{2}\Phi'(t;u)\|\nabla u(t,\cdot)\|_{H^m}^2\\ &= \Re\left(\nabla u_t(t,\cdot), \nabla u(t,\cdot)\right)\|\nabla u(t,\cdot)\|_{H^m}^2\\ &\leq E(0)^{\frac{1}{2}}\left(2\int_{\mathbb{R}^n}|\nabla u_t(t,x)|^2\,dx\right)^{\frac{1}{2}}E_m(t)\\ &\leq 2E(0)^{\frac{1}{2}}E_1(t)^{\frac{1}{2}}E_m(t) \leq 2E(0)^{\frac{1}{2}}E_m(t)^{\frac{3}{2}}\end{aligned}$$

Remark

If $|\Phi'(t;u)| < \infty$ (or $E_1(t) < \infty$), then $E_m(t) < \infty$.

Global solvability (Nishihara's method)

By applying Hardy-Littlewood-Polya inequality, we have

$$\left(\int_{\mathbb{R}^n} |\xi|^2 \left|\hat{u}_t(t,\xi)\right|^2 \, dx\right)^{\frac{1}{2}} \le \mathcal{M}^{-1}\left(\int_{\mathbb{R}^n} \mathcal{M}(|\xi|) \left|\hat{u}_t(t,\xi)\right|^2 \, d\xi\right),$$

if $\mathcal{M}(r^{1/2})$ is convex and positive. Setting

$$\mathcal{E}(t,\xi) := \frac{1}{2}\left(\Phi(t;u)|\xi|^2|\hat{u}(t,\xi)|^2 + |\hat{u}_t(t,\xi)|^2\right)$$

and

$$F(t;\mathcal{M}) := \int_{\mathbb{R}^n} \mathcal{M}(|\xi|)\mathcal{E}(t,\xi)\, d\xi.$$

Then we have

$$\frac{d}{dt}F(t;\mathcal{M}) \le F(t;\mathcal{M})\mathcal{M}^{-1}\left(\rho F(t;\mathcal{M})\right).$$

Global solvability in quasianalytic class

$$\frac{d}{dt}F(t;\mathcal{M}) \le F(t;\mathcal{M})\mathcal{M}^{-1}\left(\rho F(t;\mathcal{M})\right), \quad F(0;\mathcal{M}) < \infty,$$

$$\int_C^\infty \frac{dr}{r\mathcal{M}^{-1}(r)} = \infty \;\;\Rightarrow\;\; F(t;\mathcal{M}) < \infty, \;\; \forall t \in (0,\infty).$$

- $\mathcal{M}(r) = \exp(r)$, $\quad \int_C^\infty \frac{dr}{r\log r} = \infty$
- $\mathcal{M}(r) = \exp\left(r^{1/s}\right)$, $s > 1$, $\quad \int_C^\infty \frac{dr}{r(\log r)^s} < \infty$
- $\mathcal{M}(r) = \exp\left(\frac{r}{\log r}\right)$, $\quad \int_C^\infty \frac{dr}{r\log r\log(\log r)} = \infty$

REMARK

Smoothness property of the initial data contributes the existence time of the solution.

Prolongation of the existence time

Kirchhoff equation

$$\left(\partial_t^2 - \Phi(t;u)\Delta\right) u(t,x) = 0, \quad \Phi(t;u) = 1 + \|\nabla u(t,\cdot)\|^2$$

By the estimate of $E_1(t)$:

$$\frac{d}{dt}E_1(t) \le 2E(0)^{\frac{1}{2}}E_1(t)^{\frac{3}{2}} \Rightarrow \quad E_1(t)^{\frac{1}{2}} \le \frac{1}{E(0)^{\frac{1}{2}}\left(E(0)^{-1} - t\right)}$$

we have the following estimate of $\Phi'(t;u)$:

$$|\Phi'(t;u)| \le \frac{2}{T-t}, \quad T = E(0)^{-1}.$$

Linear hyperbolic problem

Linear wave equation with variable propagation speed:

$$\begin{cases} \left(\partial_t^2 - \Psi(t)\Delta\right) w(t,x) = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ (w(0,x), w_t(0,x)) = (w_0(x), w_1(x)), & x \in \mathbb{R}^n, \end{cases} \qquad (L)$$

where $\Psi(t) \geq 1$, $\Psi(t) \in C^m([0,T)) \cap C^0([0,T])$, $m \geq 2$.

Proposition ([Manfrin05], [H.06])

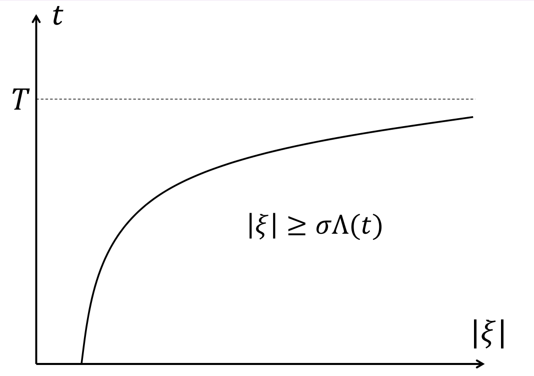
There exist positive constants C_m and ρ such that the following estimate is established for $|\xi| \geq \rho\Lambda(t)$:

$$\mathcal{E}(t,\xi) \leq \exp\left(C_m |\xi| \left(\frac{|\xi|}{\Lambda(t)}\right)^{-m}\right) \mathcal{E}(0,\xi),$$

$$\Lambda(t) = \max_{\substack{|h|=m \\ 1h_1+\cdots+mh_m=m}} \left\{ \left(\prod_h \left|\Psi^{(j)}(t)\right|^{h_j} \right)^{\frac{1}{m}} \right\}.$$

Zone in the phase space

$$\mathcal{E}(t,\xi) \le \exp\left(C_m\,|\xi|\left(\frac{|\xi|}{\Lambda(t)}\right)^{-m}\right)\mathcal{E}(0,\xi), \quad |\xi| \ge \sigma\Lambda(t)$$



Manfrin's class

For $m \in \mathbb{N}$, $\rho \geq 1$ and $\eta > 0$ we define the weight function $W_m(r;\rho)$ and the norm $G_m(f;\rho,\eta)$ by

$$W_m(r;\rho,\eta) := \left(\frac{r}{\rho}\right)^m \exp\left(\eta r \left(\frac{r}{\rho}\right)^{-m}\right),$$

$$G_m(f;\rho,\eta) := \int_{|\xi|\geq\rho} W_m(|\xi|;\rho,\eta)|\hat{f}(\xi)|^2\, d\xi.$$

Then Manfrin's class $B_\Delta^{(m)}$ is defined by

$$B_\Delta^{(m)} := \bigcup_{\eta>0} \left\{ f(x)\ ;\ \exists\{\rho_j\} \in \mathcal{L},\ \sup_j \{G_m(f;\rho_j,\eta)\} < \infty \right\},$$

where

$$\mathcal{L} := \left\{ \{\rho_j\}_{j=1}^{\infty}\ ;\ \rho_j \nearrow \infty \right\}.$$

Manfrin's class

$$W_m(r;\rho,\eta) := \left(\frac{r}{\rho}\right)^m \exp\left(\eta r \left(\frac{r}{\rho}\right)^{-m}\right),$$

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$$B_\Delta^{(m)} := \bigcup_{\eta>0}\left\{f(x)\,;\ \exists\{\rho_j\}\in\mathcal{L},\ \sup_j\{G_m(f;\rho_j,\eta)\}<\infty\right\}.$$

$$\mathcal{L} := \left\{\{\rho_j\}_{j=1}^{\infty}\,;\ \rho_j \nearrow \infty\right\}$$

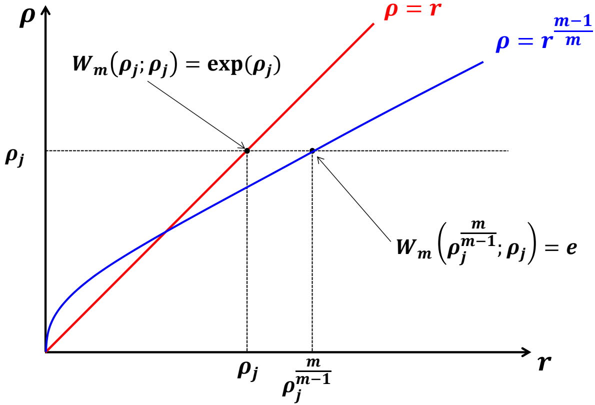
Theorem ([Manfrin 05], [H.06])

If $\nabla u_0, u_1 \in B_\Delta^{(m)}$ *for* $m \ge 2$, *then (K) has a time global classical solution satisfying*

$$\|\nabla u(t,\cdot)\|_{H^{\frac{m}{2}}} + \|u_t(t,\cdot)\|_{H^{\frac{m}{2}}} < \infty.$$

Manfrin's class

$$\boxed{W_m(r;\rho) \coloneqq W_m(r;\rho,1) := \left(\frac{r}{\rho}\right)^m \exp\left(r\left(\frac{r}{\rho}\right)^{-m}\right)}$$



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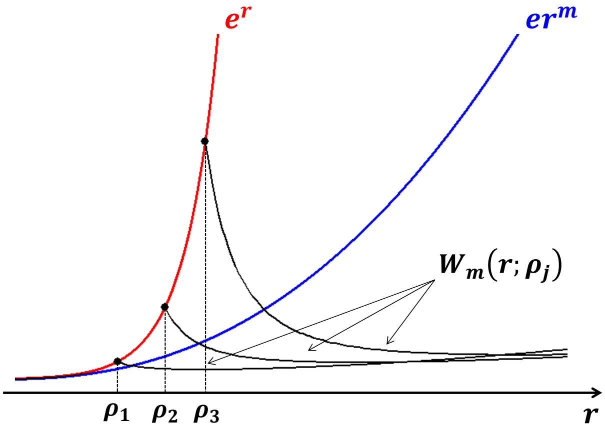

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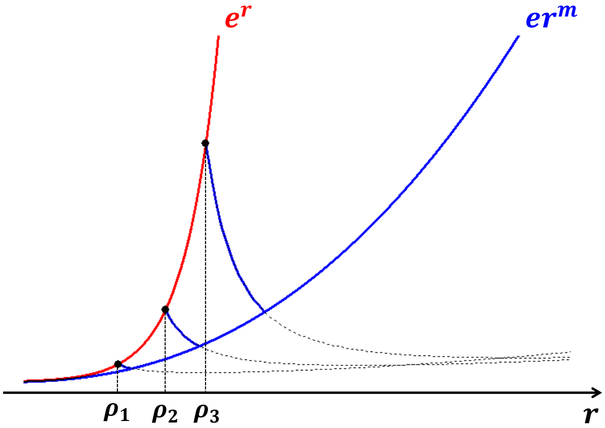

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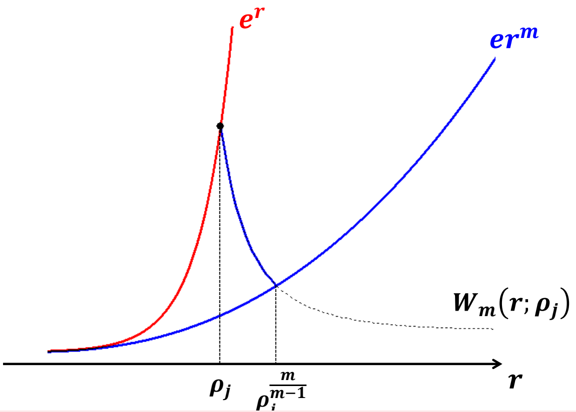
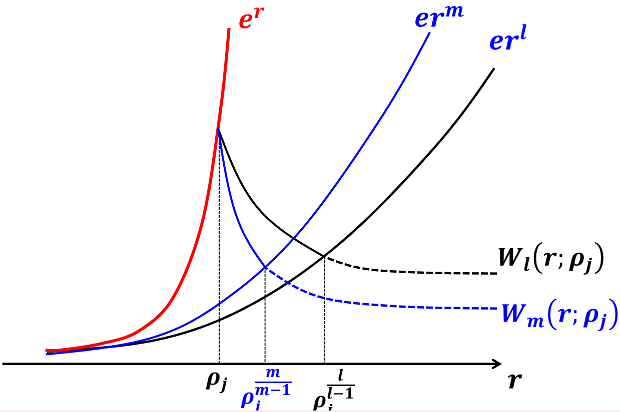


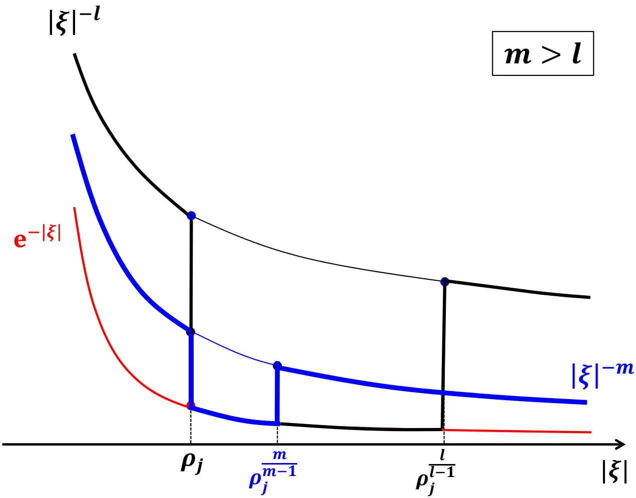
Manfrin's class

$$\boxed{W_m(r;\rho) := \left(\frac{r}{\rho}\right)^m \exp\left(r\left(\frac{r}{\rho}\right)^{-m}\right)}$$

$$\boxed{m > l}$$



Manfrin's class



Some remarks on the Manfrin's class

$$G_m(f;\rho,\eta) := \int_{|\xi|\ge\rho} \left(\frac{|\xi|}{\rho}\right)^m \exp\left(\eta|\xi|\left(\frac{|\xi|}{\rho}\right)^{-m}\right) |\hat{f}(\xi)|^2\, d\xi$$

$$B_\Delta^{(m)} := \bigcup_{\eta>0}\left\{f(x)\,;\ \exists\{\rho_j\}\in\mathcal{L},\ \sup_j\{G_m(f;\rho_j,\eta)\}<\infty\right\}$$

Proposition

(i) $B_\Delta^{(1)} \subset C^\omega$ *(realanalytic class);*

(ii) $\mathscr{Q}_N \not\subset B_\Delta^{(m)}$ *and* $B_\Delta^{(m)} \not\subset \mathscr{Q}_N$ *(quasianalytic class);*

(ii) $B_\Delta^{(m)} \subset H^{\frac{m}{2}}$ *and* $B_\Delta^{(m)} \not\subset H^{\frac{m}{2}+\varepsilon}$ *for any* $\varepsilon > 0$*;*

(iii) $B_\Delta^{(m+1)} \not\subset B_\Delta^{(m)}$.

Consideration as $m \to \infty$

The norm $G_m(f;\rho,\eta)$:

$$G_m(f;\rho,\eta) = \int_{|\xi|\ge\rho} \left(\frac{|\xi|}{\rho}\right)^m \exp\left(\frac{\eta|\xi|}{\left(\frac{|\xi|}{\rho}\right)^m}\right) |\hat{f}(\xi)|^2 \, d\xi$$

for Manfrin's class should be generalized as

$$\int_{|\xi|\ge\rho} \widetilde{\mathfrak{M}}\left(\frac{|\xi|}{\rho}\right) \exp\left(\frac{\eta|\xi|}{\mathfrak{M}\left(\frac{|\xi|}{\rho}\right)}\right) |\hat{f}(\xi)|^2 \, d\xi.$$

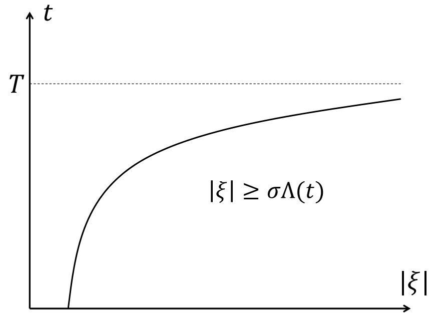
Let us consider the choice $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$ from the consequence of the properties of the linear wave equation with smooth coefficient $\Psi(t)$:

$$\left(\partial_t^2 - \Psi(t)\Delta\right) w(t,x) = 0.$$

Linear wave equation with C^m coefficient

If $\Psi(t) \geq 1$, $\Psi(t) \in C^m([0,T)) \cap C^0([0,T])$, then there exists a positive constant C_m such that

$$\mathcal{E}(t,\xi) \leq \exp\left(C_m |\xi| \left(\frac{|\xi|}{\Lambda(t)} \right)^{-m} \right) \mathcal{E}(0,\xi), \quad |\xi| \geq \sigma\Lambda(t).$$



Linear wave equation with smooth coefficient

Suppose that $\Psi(t) \in C^\infty([0,T))$ satisfies $\Psi(t) \geq 1$ and

$$\left|\Psi^{(k)}(t)\right| \leq M_k \Lambda(t)^k, \quad k = 0, 1, 2, \cdots$$

with a positive strictly increasing function $\Lambda(t)$ and a logarithmically convex sequence $\{M_k\}$; $\frac{M_k}{kM_{k-1}} \leq \frac{M_{k+1}}{(k+1)M_k}$.

Proposition ([H.10], [H.-Ishida13])

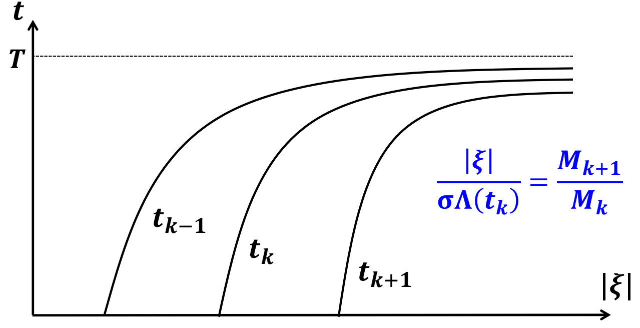
There exist positive constants σ and η such that for the sequence $\{t_k\}_{k=1}^\infty$ be defined by $\frac{|\xi|}{\sigma\Lambda(t_k)} = \frac{M_{k+1}}{M_k}$. Then the following estimates are established for $\frac{M_k}{M_{k-1}} \leq \frac{|\xi|}{\sigma\Lambda(t)} \leq \frac{M_{k+1}}{M_k}$:

$$\mathcal{E}(t,\xi) \leq \exp\left(\frac{\eta|\xi|}{\frac{1}{M_k}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^k}\right)\mathcal{E}(t_{k+1},\xi) \ \ (k = 0, 1, \cdots).$$

Linear wave equation with smooth coefficient

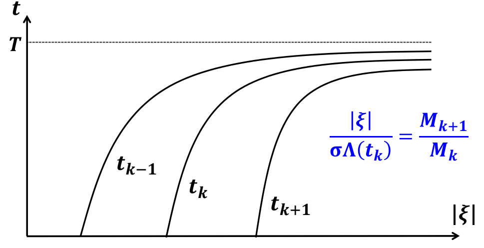
$$\left|\Psi^{(k)}(t)\right| \le M_k \Lambda(t)^k, \quad k = 0, 1, 2, \cdots$$

$$\mathcal{E}(t,\xi) \le \exp\left(\frac{\eta|\xi|}{\frac{1}{M_k}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^k}\right)\mathcal{E}(t_{k+1},\xi), \quad t_{k+1} \le t \le t_k$$



Linear wave equation with smooth coefficient

$$\mathcal{E}(t,\xi) \le \exp\left(\frac{\eta|\xi|}{\frac{1}{M_k}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^k}\right)\mathcal{E}(t_{k+1},\xi), \quad t_{k+1} \le t \le t_k$$



$$\frac{1}{M_k}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^k \ge \max\left\{\frac{1}{M_{k+1}}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^{k+1}, \frac{1}{M_{k-1}}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^{k-1}\right\}$$

Associated function of $\{M_k\}$

For a logarithmically convex sequence $\{M_k\}$ the associated function $\mathfrak{M}(r;\{M_k\})$ is defined by

$$\mathfrak{M}\left(r;\{M_k\}\right) := \sup_{k\geq 1}\left\{\frac{r^k}{M_k}\right\}, \quad r > 0.$$

Example

(i) $\mathfrak{M}\left(r;\{k!^s\}\right) \approx \exp\left(r^{\frac{1}{s}}\right)$, $s \geq 1$.

(ii) $\mathfrak{M}\left(r;\left\{\prod_{j=1}^{k}\exp\left(j^{\nu}\right)\right\}\right) \approx \exp\left(\log(1+r)^{1+\frac{1}{\nu}}\right)$, $\nu > 0$.

Choice of $\mathfrak{M}$ and $\widetilde{\mathfrak{M}}$

Generalization of the norm $G_m(f;\rho,\eta)$;

$$G_m(f;\rho,\eta) = \int_{|\xi|\geq\rho} \left(\frac{|\xi|}{\rho}\right)^m \exp\left(\frac{\eta|\xi|}{\left(\frac{|\xi|}{\rho}\right)^m}\right) |\hat{f}(\xi)|^2 \, d\xi$$

$$\Rightarrow \int_{|\xi|\geq\rho} \widetilde{\mathfrak{M}}\left(\frac{|\xi|}{\rho}\right) \exp\left(\frac{\eta|\xi|}{\mathfrak{M}\left(\frac{|\xi|}{\rho}\right)}\right) |\hat{f}(\xi)|^2 \, d\xi$$

from the consequence of the properties of the linear wave equation:

$$\mathcal{E}(t,\xi) \leq \exp\left(\frac{\eta|\xi|}{\frac{1}{M_k}\left(\frac{|\xi|}{\sigma\Lambda(t)}\right)^k}\right) \mathcal{E}(t_{k+1},\xi), \quad t_{k+1} \leq t \leq t_k.$$

Main theorem

For positive real numbers σ, ρ, η and a logarithmically convex sequence $\{M_k\}$ we define $\tilde{G}(f;\sigma,\rho,\eta,\{M_k\})$ and the class $B_\Delta(\{M_k\})$ by

$$G(f;\sigma,\rho,\eta,\{M_k\}) :=$$

$$\int_{|\xi|\geq\rho} \mathfrak{M}\left(\frac{|\xi|}{\rho};\{\sigma^k M_k\}\right) \exp\left(\frac{\eta|\xi|}{\mathfrak{M}\left(\frac{|\xi|}{\rho};\{\sigma^k k! M_k\}\right)}\right) |\hat{f}(\xi)|^2 \, d\xi,$$

$$B_\Delta(\{M_k\}) :=$$

$$\bigcup_{\eta>0,\sigma>0} \left\{ f(x)\,;\ \exists\{\rho_j\}\in\mathcal{L},\ \ \sup_j\{G(f;\sigma,\rho,\eta,\{M_k\})\} < \infty \right\},$$

where

$$\mathfrak{M}(r;\{M_k\}) := \sup_{k\geq 1}\left\{\frac{r^k}{M_k}\right\}.$$

Main theorem

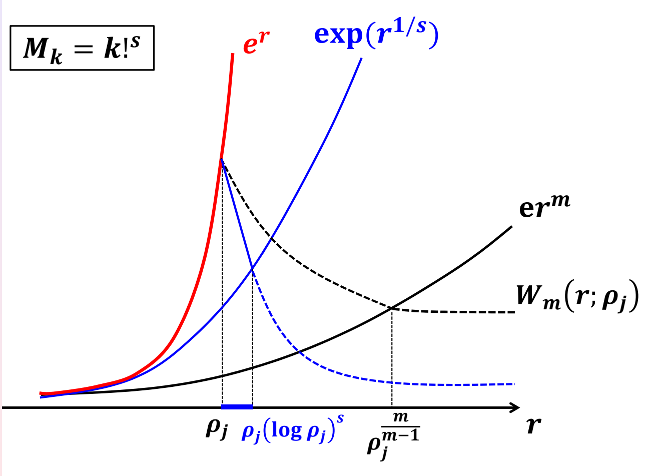
Theorem

If $\nabla u_0, u_1 \in B_\Delta(\{M_k\})$*, then the Kirchhoff equation (K) has a time global classical solution satisfying*

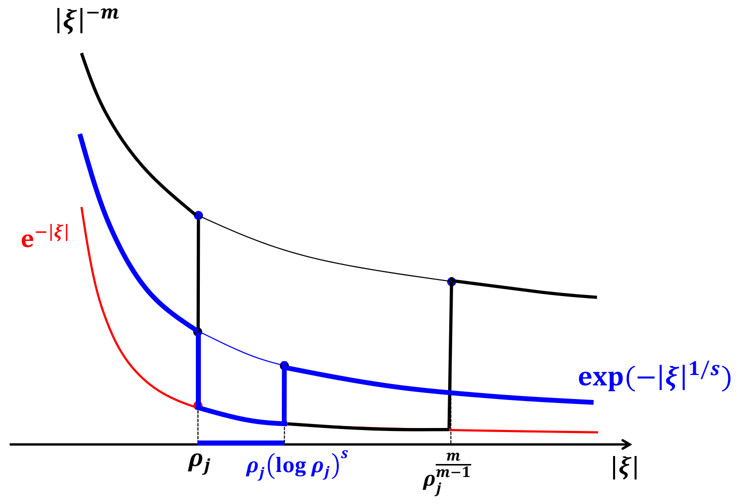
$$\int_{\mathbb{R}^n} \mathfrak{M}\left(|\xi|;\{\sigma_0^k M_k\}\right)\left(|\xi|^2|\hat{u}(t,\xi)|^2+|\hat{u}_t(t,\xi)|^2\right)\,d\xi<\infty$$

for a positive constant σ_0*.*

Examples


$M_k = k!^s$
e^r
$\exp(r^{1/s})$
er^m
$W_m(r;\rho_j)$
ρ_j
$\rho_j(\log \rho_j)^s$
$\rho_j^{\frac{m}{m-1}}$
r

Examples



Thank you for your attention!